Week 10
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 10 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Sequence, Convergence, Divergence, Squeeze Theorem, Bounded, Monotonic, Partial Sum, Linearity, Geometric Series, p-Series, Comparison Test, Algebraic Limit Laws*

Topic of the Week: Sequences and Series

Contents:

Highlight 1: 10.1 Sequences

"A *sequence* $\{a_n\}$ is an ordered collection of numbers defined by a function on a set of sequential integers. The values of $a_n = f(n)$ are called the terms of the *sequence*, and n is called the index" (Rogawski 537). One can also just think of a *sequence* as a list. A *sequence* can be finite, if it only contains a finite number of terms, or infinite if it goes on forever.

A *sequence* is said to *converge* to a limit L if, generally speaking, the further out one gets in the *sequence*, the closer the terms get to L. This concept of *convergence* is almost identical to the concept of the *convergence* of a function to a horizontal asymptote. The mathematical notations for this are

$$\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L$$

If, for a given *sequence*, such a limit L exists, the *sequence* is said to *converge*. If no such L exists, the *sequence* is said to *diverge*.

It is commonplace that the most natural way to specify the terms of a *sequence* is with a function $a_n = f(n)$. In this case, the difference between the *sequence* and the function is that the function is defined on it's entire (real) range, while the *sequence* is only defined on discrete, counting values of n. Intuitively, if the function *converges* to some asymptote, the *sequence* also *converges* to that value:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

The usefulness of this seemingly obvious statement is that all of one's tools for functional *convergence* translate directly over to *sequences*: *convergence* tests, *comparison tests*, and the like.

A special type of *sequence* is the *geometric sequence*, which is any *sequence* in the form $a_n = c * r^n$. The rules for *convergence* of a *geometric sequence* are as follows:

$$\lim_{n\to\infty} c * r^n = \begin{cases} 0 & if \quad 0 \le r < 1 \\ c & if \quad r = 1 \\ \infty & if \quad r > 1 \end{cases}$$

The *Squeeze Theorem* translates to *sequences*, as do the *algebraic limit laws*:

THEOREM 2 Limit Laws for Sequences Assume that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L, \qquad \lim_{n\to\infty} b_n = M$$

Then

(i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$

(ii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$

(iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$ if $M \neq 0$

(iv) $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$ for any constant c

(Rogawski 541).

It is also true that limits can commute with continuous functions, such that

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

A *sequence* is said to be *bounded from above* if there is some single number that is greater than every number in the *sequence*. A *sequence* is said to be *bounded from below* if there is some single number that is less than every number in the *sequence*. A *sequence* is simply called *bounded* if it is *bounded from above* and *from below*, and a *sequence* is called *unbounded* if it is not *bounded*. It is easy to prove *boundedness* from *convergence*. But we are often interested in proving *convergence*, and to prove *convergence* from *boundedness*, we need another concept: *monotonicity*.

A *monotonic increasing sequence* is one who's terms are always increasing. A *monotonic decreasing sequence* is one who's terms are always decreasing.

If $\{a_n\}$ is *bounded above* by M and is *increasing monotonic*, then $\{a_n\}$ *converges* and $\lim_{n\to\infty} a_n \leq M$. If $\{a_n\}$ is *bounded below* by m and is *decreasing monotonic*, then $\{a_n\}$ *converges* and $\lim_{n\to\infty} a_n \geq m$ (Rogawski 544).

Highlight 2: 10.2 Infinite Series

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A *series* is the sum of a *sequence*. Where a *sequence* might look like this
1, 2, 4, 8, 16, 32, 64, …
a series will look like this
1+2+4+8+16+32+64+…

Typically, the *series* that interest us in mathematics are infinite, meaning they consist in a sum of an infinite number of terms. We call the n^{th} *partial sum* (S_n) of an *infinite series* the sum of the first n terms. For example, the 4^{th} *partial sum* of the above *series* is $S_4 = 1+2+4+8 = 15$. We are often interested in proving the *convergence* of *infinite series*, and a very useful theorem is that if the *sequence* of *partial sums* of a *series converges*, then the *infinite series* also *converges*. In math notation,

$$\sum_{n=k}^{\infty} a_n \ converges\ to\ S\ if\ \lim_{n\to\infty} S_n = S,$$

$$in\ which\ case\ we\ say\ S = \sum_{n=k}^{\infty} a_n.$$

$$Otherwise, the\ series\ \text{diverges}\ (Rogawski\ 549).$$

Convergent infinite series satisfy the following *linearity* property:

THEOREM 1 Linearity of Infinite Series If $\sum a_n$ and $\sum b_n$ converge, then $\sum (a_n + b_n)$, $\sum (a_n - b_n)$, and $\sum c a_n$ also converge, the latter for any constant c. Furthermore,

$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$

$$\sum (a_n - b_n) = \sum a_n - \sum b_n$$

$$\sum c a_n = c \sum a_n \qquad (c \text{ any constant})$$

(Rogawski 551).

Formula for the sum of the first N terms of the *geometric series* $\sum_{n=0}^{\infty} c * r^n$:

$$S_N = c + cr + cr^2 + cr^3 + \ldots + cr^N = \frac{c(1-r^{N+1})}{1-r}$$

And the entire *series* (infinite sum):

$$\sum_{n=0}^{\infty} c * r^n = c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1-r}, provided\ -1 < r < 1.$$

If r, in the above case, has an absolute value greater than or equal to 1, then the *infinite series diverges*.

One might have guessed the following result relating the *convergence* of *sequences* and the *divergence* of the associated *series*: If a *sequence converges* to anything other than 0, then the *series* you get from summing it all up must *diverge*. Mathematically,

$$If\ \lim_{n\to\infty} a_n \neq 0, \quad then \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Highlight 3: 10.3 Convergence of Series

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This chapter gives a variety of *tests* to determine the *convergence* of a *series*.

THEOREM 1 Partial Sum Theorem for Positive Series If $\sum_{n=1}^{\infty} a_n$ is a positive series, then either

(i) The partial sums S_N are bounded above. In this case, $\sum_{n=1}^{\infty} a_n$ converges. Or,

(ii) The partial sums S_N are not bounded above. In this case, $\sum_{n=1}^{\infty} a_n$ diverges.

(Rogawski 561)

This theorem translates the problem of determining the *convergence* of a *series* into the simpler problem of determining the *convergence* of a *sequence*. To every *sequence* corresponds a *series*, which is the sum of the terms of the *sequence*. But to every *series* corresponds another *sequence*, which is the rolling *partial sums* of the terms of the *series*. For example, we may have an original *sequence*,

1, 2, 4, 8, 16, 32, …

the corresponding *series*,

1+2+4+8+16+32+…

and the resultant *sequence* of *partial sums*,

1, 3, 7, 15, 31, 63, …

The above theorem works from the *convergence* of *monotonic bounded sequences*. Because we are limiting ourselves to *series* consisting entirely of positive terms, each subsequent *partial sum* must be larger than the last. Accordingly, the *sequence* of *partial sums* must *monotonically increasing* (always increasing). Thus, when we add the condition that the *sequence* of *partial sums* is *bounded above*, then by the result at the end of 10.2, it must be that the *sequence* of *partial sums converges*. We also know that if a *sequence* of *partial sums converges*, it must also be the case that the infinite *series* in question *converges*.

THEOREM 2 Integral Test Let $a_n = f(n)$, where f is a positive, decreasing, and continuous function of x for $x \geq 1$.

(i) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

(Rogawski 561)

This result is a consequence of the *comparison test* for *convergence* and *divergence* of improper integrals. When one considers the fact that the terms of the *sequence* $\{a_n\}$ are just a subset of the function f(n), then one will see that the *sequence* $\{a_n\}$ is always less than or equal to the function f(n) wherever it is defined. Accordingly, if the area under the function f(n) between 1 and ∞ *converges* to some value, then it must be the case than the "area under the *sequence*" (the sum of the *sequence*, i.e. the *series*) also *converges*, since the second area is *bounded* by the first. The reverse logic applies for the *divergence* of a *series*.

THEOREM 3 Convergence of *p*-Series The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

(Rogawski 562)

This result is interesting and useful in itself and may be used to prove the *divergence* of the harmonic *series*. Like the corresponding theorem for *p-power* integral *convergence*, this theorem can be used in conjunction with the integral *test* and with the following direct *comparison test* to prove the *convergence* or *divergence* of other *series*, even if they do not fit into the exact form $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

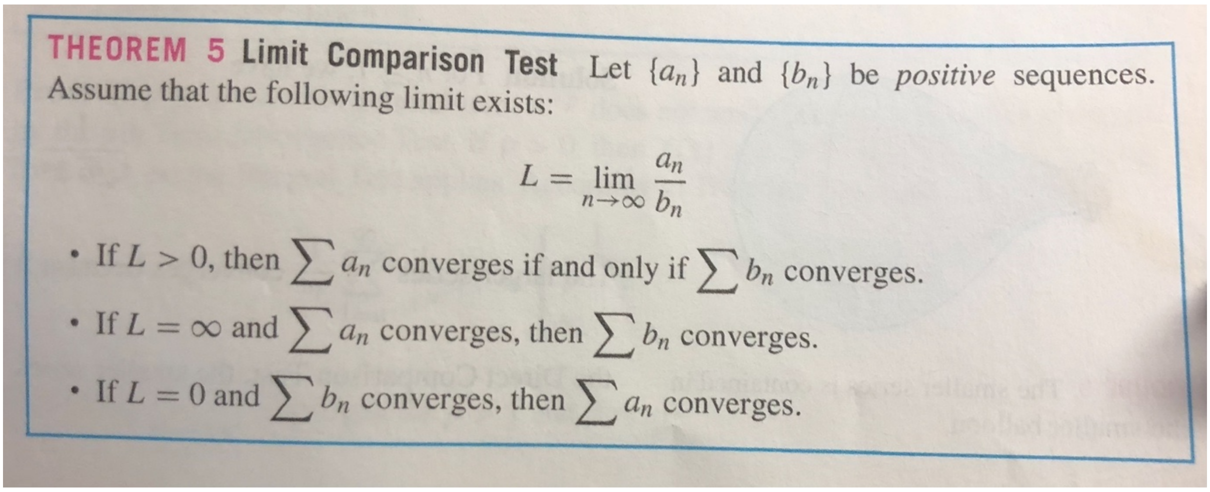
THEOREM 4 Direct Comparison Test
Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

(Rogawski 563)

This theorem corresponds to the direct *comparison test* for integrals. The result is intuitive enough. If each term in the first *sequence* is smaller than each term in the second *sequence*, and the second *series converges*, than the first *series* must also *converge*. The reverse logic applies for *divergence*.



THEOREM 5 Limit Comparison Test Let $\{a_n\}$ and $\{b_n\}$ be *positive* sequences. Assume that the following limit exists:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

- If $L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.
- If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(Rogawski 564)

The intuition in this theorem is to rearrange the limit statement $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ to something more like $a_n \approx L * b_n$ for the L>0 case. Stated like this, one will see that $\sum a_n$ is approximately a multiple of $\sum b_n$, so it is natural to think that if one *converges*, then the other will also *converge*. If L=∞, then it must be that a_n is much larger than b_n for large n. Accordingly, if $\sum a_n$ *converges*, then $\sum b_n$ will also have to *converge*. Lastly, if L=0, then it must be that b_n is much larger than a_n for large n. Accordingly, if $\sum b_n$ *converges*, then $\sum a_n$ will also have to *converge* (Rogawski 565).

Check Your Learning

1. Determine the limit or state the *divergence* of the *sequence* $a_n = \frac{e^n}{2^n}$.

(Rogawski 547)

2. Find the sum or state the *divergence* of the *series* $\sum_{n=0}^{\infty} \frac{8+2^n}{5^n}$.

(Rogawski 558)

3. Use the Direct *Comparison Test* to show *divergence*/*convergence* of $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}+2^n}$.

(Rogawski 567)

Things you may Struggle With

1. *Sequences and Functions*- One will note the striking similarity between the *convergence* properties of functions and those of *sequences*. This is because *sequences* technically are functions; they are just functions that are defined on a discrete domain rather than on the continuous real number line. Thus, there is nothing new in chapter 10.1; this chapter simply states special cases of more general functional *convergence* theorems.

2. *Symbols vs.* Intuition- All the tests in the *convergence* chapter are intuitive. Mathematical symbols do not always convey the right intuition, so focus instead on their geometric meaning. Draw *pictures*. Plot the *sequences* to be summed and the functions to be integrated. Depending on whether one is asked to prove *convergence* or *divergence*, the picture drawn will help one recognize what kind of *sequence*/function will be useful for a *comparison test*.

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. *Diverges*

2. $S = \frac{35}{3}$

3. $\frac{1}{n^{1/3}+2^n} \leq \left(\frac{1}{2}\right)^n$, so the *series converges*.

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.