Week 11
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 11 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Absolute Convergence, Conditional Convergence, Divergence, Alternating Series, Ratio Test, Root Test*

Topic of the Week: Convergence Tests

Contents:

Highlight 1: 10.4 Absolute and Conditional Convergence

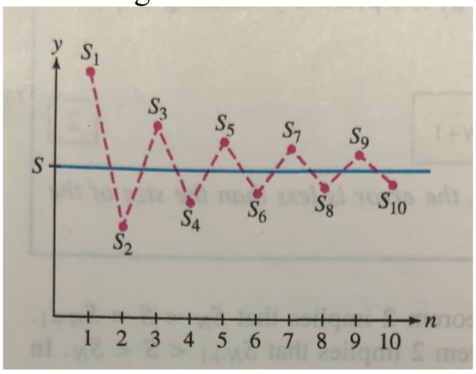
A series *converges absolutely* if the sum of the absolute values of its terms *converges*. That is, $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ *converges*. This statement serves as both a definition and a theorem. The above criterion is useful for proving the *convergence* of *alternating series* — ones where each subsequent term alternates between positive and negative.

This definition prompts a related definition of *conditional convergence*. We say that a series *converges conditionally* if it *converges*, but the sum of the absolute values of its terms *diverges*. That is, $\sum_{n=1}^{\infty} a_n$ *converges conditionally* if $\sum_{n=1}^{\infty} a_n$ *converges* but $\sum_{n=1}^{\infty} |a_n|$ *diverges* (Rogawski 570). This conditional is merely a definition; nothing is added to our mathematical knowledge by stating it. However, it will allow us to refer to some important situations that will expand our knowledge.

The *alternating series test* states that for any positive decreasing sequence $\{b_n\}$ that *converges* to zero, $\left(b_1 > b_2 > b_3 > \ldots > 0,\ \lim_{n\to\infty} b_n = 0\right)$, the following series *converges*:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

In addition, that sum is bounded by 0 on the bottom and b_1 on the top (Rogawski 571). The sum is also bounded by every even and odd term. One can form a mental image of the partial sums of such a series, like the following:



(Rogawski 571)

As you can see, the partial sums of an *alternating series* form a zig-zag pattern that narrows in on S, the actual infinite sum. Amazingly, this broad statement is true of every *alternating series* meeting the above description.

Consider the following example dealing with the *convergence* of *alternating series*.

We are asked to prove that the *alternating* version of the harmonic series *converges conditionally*:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

One will recall that the harmonic series itself *diverges*, because it is p-power type sum of degree 1. Furthermore, observe that the ordinary harmonic series consists of all positive terms that are always growing smaller, approaching zero. Therefore, the *alternating harmonic series* meets all criteria to apply the *alternating series test* and conclude *convergence*. And an *alternating series* that *converges* but whose corresponding absolute values series *diverges* is said to *converge conditionally*. Q.E.D.

Highlight 2: 10.5 Ratio and Root Tests

Sometimes, one is able to prove the *convergence* or *divergence* of a series just from examining what happens to the ratio of subsequent terms as the index increases. The *ratio test* is stated formally as follows:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$If\ \rho < 1, then \sum_{n=1}^{\infty} a_n\ converges\ absolutely.$$

$$If\ \rho > 1, then \sum_{n=1}^{\infty} a_n\ diverges.$$

The test is inconclusive in the case that ρ is exactly equal to 1 (Rogawski 575). From the structure of the ratio limit, one can see that series with terms that increase in magnitude are automatically ruled out as *divergent*, but so are series with terms that don't "get small fast enough." The fine line between "getting small fast enough" and "not getting small fast enough" is what the *ratio test* enables you determine.

For example, prove that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *converges* (Rogawski 576). Observe that

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1} n!}{(n+1)!\, 2^n}\right| = \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0$$

Since $\rho = 0 < 1$, by the *ratio test*, we conclude that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *converges*.

Similar to the *ratio test* is the *root test*. The *root test* is stated formally as follows:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$If\ L < 1, then \sum_{n=1}^{\infty} a_n\ converges\ absolutely.$$

$$If\ L > 1, then \sum_{n=1}^{\infty} a_n\ diverges.$$

This test is inconclusive if L is exactly equal to 1 (Rogawski 577). The *root test* may seem a rather random and therefore useless tool for determining *convergence*. However, there is a certain class of series for which the *root test* is the best means to prove *convergence*. If the terms of the series are of the form of $f(n)^{g(n)}$, it may be the case that the terms will be greatly simplified by taking the n^{th} root, in which case the root test is your friend.

For example, does $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ *converge* (Rogawski 577)? We notice that

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2} < 1$$

Therefore, since L<1, by the *root test*, the series *converges absolutely*.

The last few sections have introduced test after test for determining *convergence*. It can be difficult to know which test to apply in a given circumstance. Sometimes, the *root test* or *ratio test* will just jump out as the obvious choice, but sometimes it is unclear which test to use. The text contains a helpful control flow outline of the order in which to try tests (Rogawski 577). I will summarize the outline here.

1. Check to see if the terms themselves are approaching anything other than zero. If they are, the series *diverges*, 'nuff said.

2. If you're dealing with a positive series, one of the following tests may be of use: Direct Comparison Test, Limit Comparison Test, *Ratio Test*, *Root Test*, Integral Test.

The text contains more heuristics for selecting a fitting test (Rogawski 578). Also, see previous tutoring resources (Week 10- Sequences and Series).

3. If you're dealing with an alternating series, try using the *Alternating Series Test* or the *Absolute Convergence* criterion to prove *convergence*.

Check Your Learning

1. Does $\sum_{n=1}^{\infty} \frac{(-1)^n e^{-n}}{n^2}$ converge absolutely, conditionally, or not at all?

2. Does $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converge or diverge?

Things you may Struggle With

1. Where to start- It can feel vary abstract and bewildering when one is tossed an infinite series and asked whether it converges. The key to getting started is to have the tests memorized, to understand what the theorems are and what they mean. If you have the tests in mind and don't know which to choose, just pick one you feel good about. You might not pick the right one on the first try, but as you try a couple, you'll get quite good at it. Concretely, simply write down the beginning of the test, substitute in the specifics of the problem at hand, and do algebra until you get a limit you can evaluate.

2. Symbology. Be sure you remember what the symbols mean. Here is a summary:

- n — an index (1, 2, 3, …)
- N — a particular point in the index
- $\{a_n\}$ — a sequence
- a_n — the n^{th} term of a sequence
- S — a (convergent) infinite sum (of a series)
- S_n — the n^{th} partial sum of a series
- ρ, L — limits

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. Converges absolutely

2. Converges absolutely

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.