Week 12
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 12 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Power Series, Center, Interval of Convergence, Radius of Convergence, Geometric Series, Term-by-Term, Power Series Expansion*

Topic of the Week: Power Series

Contents:

Highlight: 10.6 Power Series

In mathematics, there is a special class of series called *power series*. A *power series* is a series whose terms take the form of some power of x, and the powers of x increase with the index of the series. In math notation, a *power series* is given in its general form:

$$F(x) = \sum_{n=1}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

In this statement of a *power series*, an allowance is made for the series to be "off-*center*" by the fixed distance "c" (Rogawski 580). A *power series* may converge sometimes and diverge other times depending on the value one sets for x. The range of values of x for which the series converges is called the *interval of convergence*. The *interval of convergence* takes the form of $c \pm R$, where R is some length, called the *radius of convergence*. The form of this interval derives from the fact that the constant "c" is the *center* of the *power series*.

Every *power series* is said to have such a *radius of convergence* R. The series converges absolutely when $|x - c| < R$ and diverges when $|x - c| > R$. In other words, if R is infinite, the series converges absolutely for all values of x (Rogawski 581).

For example, what is the *radius of convergence* of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$? Applying the ratio test,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}2^n}{2^{n+1}x^n}\right| = \lim_{n\to\infty} \frac{1}{2}|x| = \frac{1}{2}|x|$$

Setting up the criterion for convergence using the ratio test,

$$\rho = \frac{1}{2}|x| < 1 \Longrightarrow |x| < 2$$

In other words, the series converges whenever the x is between -2 and 2. Therefore, 2 is the *radius of convergence* (Rogawski 582).

A subset of *power series* are *geometric series*, that is, series for which consecutive terms always form the same ratio. One will recall the famous theorem concerning *geometric series* from algebra:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \qquad for\ |x| < 1$$

This formula can be used both forwards and backwards — that is, either for evaluating a *geometric series* or for translating a rational function into the form of a *geometric series*.

Power series are of interest to mathematicians because, as it turns out, many functions can be defined alternatively as a *power series expansion* (sinusoidal, exponential, logarithmic). In fact, some functions are only definable as infinite series. The advantage to treating of functions as series is that it is often easier to work with a sum of powers of x than with one complex function. Thus, *power series* are a tool for simplifying functions to a form we can manipulate.

For instance, one common application of *power series* is to differentiate or integrate functions that would be difficult to deal with otherwise. The basic principle is as follows: One can differentiate a function simply by doing *term-by-term* differentiation on the *power series expansion* of said function. The same applies for integration. The advantage of such a technique is that it is very easy to differentiate/integrate powers of x and sums of those powers.

A formal statement of *term-by-term* differentiation and integration is (Rogawski 584)

THEOREM 2 Term-by-Term Differentiation and Integration Assume that

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R > 0$. Then F is differentiable on $(c-R, c+R)$. Furthermore, we can integrate and differentiate term by term. For $x \in (c-R, c+R)$,

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

$$\int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} \qquad (A \text{ any constant})$$

For both the derivative series and the integral series the radius of convergence is also R.

The meaning of *term-by-term* is that one simply integrates each term of the series expansion of the function. This sounds like more work, but one is really only integrating one term, the general form of all terms. The enormous utility of this approach lies in the fact that every function can be expressed as a *power series expansion*. The implication is that the problem of integrating any function can be reduced to the far simpler problem of integrating a polynomial.

Term-by-term integration is useful not only for finding exact mathematical formulas but also for getting quick, approximate results. These are obtained by taking a partial sum of the terms of the *power series expansion* of the function. One can take as large or as small a partial sum as one's tolerance for inaccuracy will permit.

A sophisticated implementation of these several principles is found in the following example.

Suppose we are asked to prove the following, for $|x| < 1$,

$$tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$$

Remember from the section on trigonometric integrals that

$$tan^{-1}x = \int \frac{dx}{1+x^2}$$

We prepare to take advantage of the formula for the sum of a *geometric series* by manipulating the integrand ever so slightly

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Now, applying the formula for the sum of a *geometric series*, backwards,

$$\frac{1}{1-(-x^2)} = \sum_{n=1}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + \cdots$$

Now, taking advantage of the fact that we may integrate a function by integrating its *power series expansion*, we can put all of the above pieces together:

$$tan^{-1}x = \int \frac{dx}{1+x^2} = \int \frac{dx}{1-(-x^2)} = \int \sum_{n=1}^{\infty} (-x^2)^n \, dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$$

Which was to be demonstrated (Rogawski 585).

A common method in the field of differential equations is to express the functional solution as a *power series*. This method works by taking advantage of the aforementioned easy differentiation and integration of *power series*. I recommend [this video] for an example of how to use *power series* to solve a differential equation encountered earlier in the course.

For your reference, the *power series expansion* of the exponential function is included below.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Check Your Learning

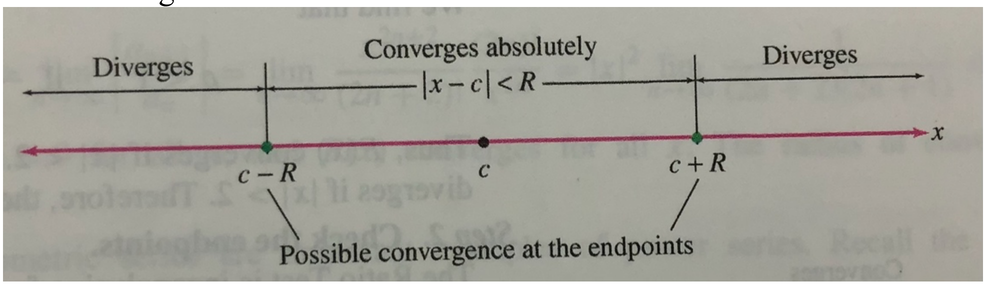
1. Find the *interval of convergence* of $\sum_{n=0}^{\infty} nx^n$ (Rogawski 589).

2. Explain how one might prove that for |x| < 1, $\int \frac{dx}{x^4+1} = C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots$

Things you may Struggle With

1. Relation to previous chapters– A *power series* is just one particular type of infinite series. \ All the techniques for determining convergence from previous chapters apply to *power series* as well. It is not uncommon that you will need to use the ratio or root test in order to find the *interval of convergence* for a *power series*.

2. *Interval of convergence*– The mental picture corresponding to *the radius of convergence* is the following:



(Rogawski 581)

Observe that the *power series* diverges for all values of x that do not fall within a radius R of the center "c." Of course, here also lies the intuition of why an infinite R implies convergence for all values of x. If one wishes to know whether the *power series* converges at the endpoints of the interval, one will need to actually plug in each value to test convergence.

Thanks for checking out these weekly resources!

Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. (-1, 1)

2. Manipulate the integrand to the form $\frac{1}{1-u}$. Apply the formula for the sum of a *geometric series*. Integrate *term-by-term*.

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.