Week 13
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 13 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Approximation, Agreement to Order n, Taylor Polynomial, Centeredness, Maclaurin Polynomial, Error Bound*

Topic of the Week: Taylor Polynomials

Contents:

Highlight: 10.7 Taylor Polynomials

In the last chapter, we examined power series — infinite series whose terms consist of ever-increasing powers of x. In this chapter we will consider a modification on power series: *Taylor polynomials*.

Before we define what a *Taylor polynomial* is, it is necessary to establish some vocabulary. First, you may be familiar with the concept that one can *approximate* one function with a simpler function. *Approximations* are only good around a point. For example, for points within a narrow radius of zero, $y = \sin(x)$ can be very well *approximated* by the line $y = x$. However, outside of that narrow window, $y = x$ is a horrible *approximation* for $y = \sin(x)$. Loosely speaking, we call the point in the middle of that radius of *approximation* the *center*, and we say that the *approximation* is *centered* at that point (zero in our example).

What makes one *approximation* better than another? One criterion is to count how many derivatives of the *approximation* agree with the corresponding derivatives of the function to be *approximated*. When the first n derivatives of the *approximation*, evaluated at the *center*, are equal to the first n derivatives of the function to be *approximated*, evaluated at the *center*, we say that the original function and its *approximation agree to order n* and that the *approximation approximates* the original function *to order n* (Rogawski 592).

We are now ready to define a *Taylor polynomial*. For a given function f, and a value in its domain "a," a *Taylor polynomial* is a series of the following form (Rogawski 593):

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

What can we say about the *Taylor polynomial*? First, note that this series is finite; it stops at "n." Because these n + 1 terms of the *Taylor polynomial* are constructed using the derivatives of f, one might suspect that, using our previous definition, a *Taylor polynomial agrees with f to order n*. This intuition would be correct. We also say that a *Taylor polynomial approximates f to order n* at "a." In this case, "a" is the *center* of the *approximation*: the point around which the *approximation* is valid, within a radius. On this note, we call a *Taylor polynomial* a *Maclaurin polynomial* when a = 0 (Rogawski 593).

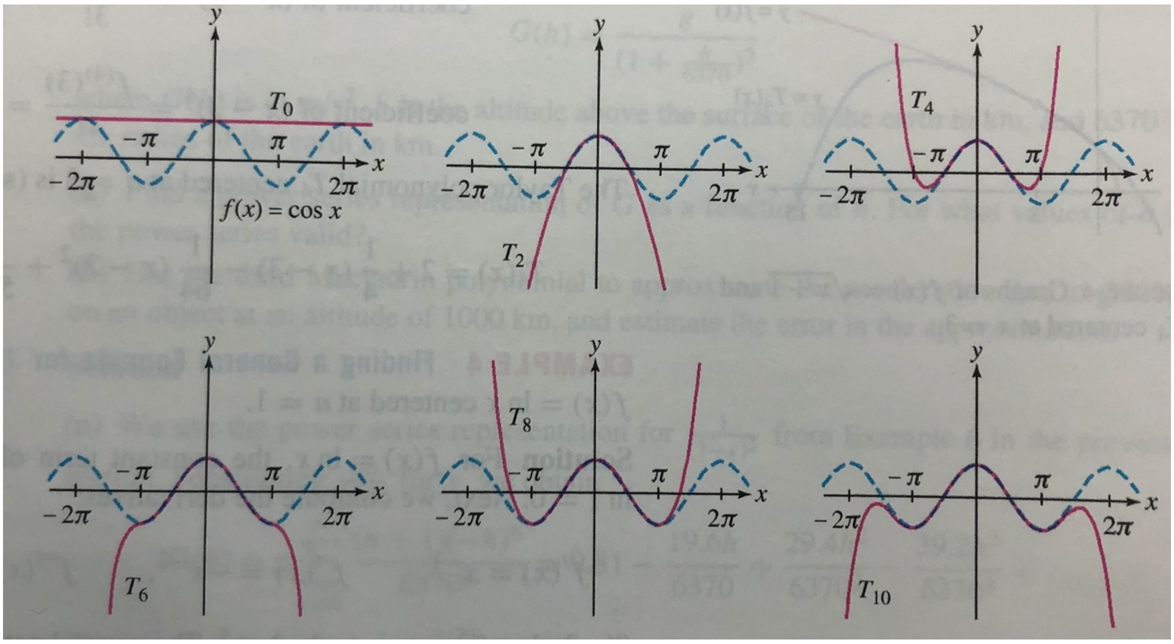
Let's consider an example. What *Maclaurin polynomial* (*Taylor polynomial centered* at 0) *approximates* the exponential function to *order* 5? We need the first 5 derivatives of the exponential function, each evaluated at zero. The slope of e^x at 0 is 1, and all subsequent derivatives of e^x are just e^x again. Therefore, the first 5 derivatives of the exponential function are each 1. Therefore, the 5^{th} *order Maclaurin approximation* is as follows:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

Notice that the derivative factors just become implied 1s, and the *centeredness* at zero causes the (x – a) terms to just become x, and all of a sudden, the *Taylor approximation* looks very similar to the series expansion for e^x you may have learned in algebra or calculus 1.

Taylor polynomials are of computational use, because they reduce the problem of finding the value of a complicated function to finding the value of a simpler function, as evidenced in the exponential case above.

The process for finding the *nth order Taylor polynomials* for the natural logarithm, square root, and trigonometric functions is very similar. Essentially, as long as we can calculate derivatives, we can get *Taylor polynomial approximations* of any *order* we desire.



Observe the visualization of *Maclaurin polynomial approximations* of the cosine function of various *orders* (Rogawski 596). Notice that with higher *orders*, the radius on which the *approximation* is valid expands. Notice also that it takes a lot of terms to expand the radius of *approximation* very far. Even with 10 terms in our series *approximation*, we cannot accurately *approximate* cosine outside the (-3π/2, 3π/2) window. This behavior is typical of *Taylor polynomial approximations*. It should serve as a reminder not to get carried away with one's *approximations* and to remember that an *approximation* is only valid within a specific radius, outside of which it is ludicrous.

$|f(x) - T_n(x)|$ is the *error* of the *Taylor polynomial approximation*. If we know of a K such that $\left|f^{(n+1)}(u)\right| \leq K, \ \forall\, u \in \ [a, x]$, then we can put the following *bound on the error* (Rogawski 596):

$$|f(x) - T_n(x)| \leq K\frac{|x - a|^{n+1}}{(n+1)!}$$

For example, suppose we are asked to find a *bound on the error* of the 3rd *order Taylor polynomial approximation*, *centered* at 1, of the natural logarithm at 1.5 (Rogawski 596). We will not actually need to construct the *Taylor polynomial*, as the *error bound* formula does not require it. We will, however, need the fourth derivative of the natural logarithm. As requested, we will take a = 1 as our *center*.

The fourth derivative of ln(x) is $-6x^{-4}$, the absolute value of which is $6x^{-4}$. This function is decreasing on the range [1, 1.5]. Accordingly, we take $6(1)^{-4} = 6$ as our K, which is always greater than the absolute value of the fourth derivative on the range [1, 1.5]. The pieces of the *error bound* formula are coming together. Observe

$$|ln(1.5) - T_3(1.5)| \leq 6\frac{|1.5 - 1|^{3+1}}{(3+1)!} \approx 0.0156$$

Check Your Learning

1. Find the *Taylor polynomial centered* at 0 to *order* 3 to *approximate* tan(x) (Rogawski 600).

2. For $f(x) = e^{-x}$ and $T_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$, show that $\frac{x^4}{24}$ is an *error bound* for all $x \geq 0$ (Rogawski 601).

Things you may Struggle With

1. Attention to detail- When dealing with *Taylor polynomials*, it is very easy to mistake an nth derivative for a power, reverse a sign, or drop a factorial. Every symbol in the definition of the *Taylor polynomial* is crucial to its meaning, so it is important that one write every mark intentionally and with understanding.

2. Evaluating derivatives- There is an important difference between evaluating a derivative at a particular point and differentiating a function evaluated at a particular point. This chapter calls for the former. We must, at first, leave variables as they are and differentiate, only plugging in concrete values at the end. In other words, we can't skip the hard work of differentiating. We will likely get the wrong answer if we plug in concrete values first and then try to differentiate.

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. $T_3(x) = x + \frac{x^3}{3}$

2. $\frac{d^4}{dx^4} e^{-x} = e^{-x}$. Since e^{-x} is decreasing for all $x \geq 0$, $K = e^0 = 1$. Therefore, a bound for the *Taylor approximation* is $1 * \frac{|x-0|^{3+1}}{(3+1)!} = \frac{x^4}{24}$.

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.