Week 14
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 14 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Taylor Series Expansion, Maclaurin Series, Binomial Series, Euler's Formula*

Topic of the Week: Taylor Series

Contents:

Highlight: 10.8 Taylor Series

The concept of a Taylor polynomial from the last chapter can be expanded into the concept of a *Taylor series*. A *Taylor series* is simply a Taylor polynomial with an infinite number of terms. In other words,

$$T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Like in the case of Taylor polynomials, we use "*Maclaurin*" to denote a *Taylor series* centered at 0 (that is c = 0). As it turns out, it can be proved that if a function can be expressed by a power series, then that series must have the above form (Rogawski 603). Furthermore, so long as a function is infinitely differentiable at some point, it has a *Taylor series expansion* at that point. Not every function falls into this category, but it does include the vast majority of the functions we deal with in lower math and in everyday life.

We may state this result formally as follows

THEOREM 2 Let $I = (c - R, c + R)$, where $R > 0$, and assume that f is infinitely differentiable on I. Suppose there exists $K > 0$ such that all derivatives of f are bounded by K on I:

$$|f^{(k)}(x)| \le K \qquad \text{for all} \quad k \ge 0 \quad \text{and} \quad x \in I$$

Then f is represented by its Taylor series in I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n \qquad \text{for all} \quad x \in I$$

(Rogawski 603).

In plain English, this theorem means that, on a given interval, so long as f is infinitely differentiable, and the higher derivatives of f stay finite, then f is equal to its *Taylor series expansion*. This statement is only true at the values of x in the prespecified interval, but for many important functions (square root, trigonometric, exponential, logarithmic) this interval can be made arbitrarily large. The effect, therefore, is that we can represent most of the functions we are interested in with a *Taylor series expansion*.

For example, what is a *Maclaurin series expansion* for the sine function? Well, the first, second, third, and fourth derivatives of sine are cosine, -sine, -cosine, and sine, after which the cycle repeats. Evaluated at 0 (as they are for a *Maclaurin series*), these derivatives come to 1, 0, -1, and 0, and the pattern continues for higher orders. The alternating 0s cause the even-powered terms of the series to disappear, so the remaining terms form the following *Maclaurin series expansion* for sine:

$$T(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Now, since all of sine's derivatives are either cosine, -sine, -cosine, or sine, they are all bounded by K = 1. Therefore, by the above theorem, we can say that sin(x) = T(x) for all x on (-R, R). But since we never specified an R, and because sine is differentiable on its entire domain, we can choose any R we like. In short, the sin(x) = T(x) for all x (Rogawski 605).

Once you know the *Taylor series* for one function, it is often possible to find the *Taylor series* for another, related function. For example, if one knows the *Taylor series* for e^x, one can easily find the *Taylor series* for e^{-x^2}, simply by substituting in $-x^2$ for x in every term of the original series (Rogawski 606).

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \ldots$$

As with Taylor polynomials, if it's hard to integrate the original function, we can just integrate the *Taylor series expansion* instead. If a function is equal to its *Taylor series expansion*, it is possible to get an exact result for the antiderivative just from integrating its *Taylor series*.

We can combine the above two principles to integrate functions for which there appears to be no analytic solution. For example, what is the area under the curve $\sin(x^2)$ between 0 and 1?

Remember that

$$sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \Rightarrow sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

So,

$$\int_0^1 sin(x^2)dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{4n+2} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{4n+3}\right) = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \ldots$$

What's more, we can put a bound on our error using the error bound formula as follows: It must be the case that

$$\left| \int_0^1 sin(x^2)dx - \left(\frac{1}{3} - \frac{1}{42} + \frac{1}{1320}\right) \right| < \frac{1}{75600} = 1.3 * 10^{-5}$$

Thus, we can say, with a percentage error less than 0.005% that

$$\int_0^1 sin(x^2)dx = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} = 0.31028$$

(Rogawski 607).

A special series to figure out the coefficients of a *binomial expansion* is as follows,

$$(x+y)^N = \sum_{n=0}^{N} \binom{N}{n} x^n y^{N-n}$$

The Taylor series of sine, cosine, and the exponential function are also used to prove *Euler's formula* for all complex numbers z:

$$e^{iz} = cos(z) + i\, sin(z)$$

In other words, with *Taylor series*, it even becomes easy to deal with something so abstract as exponentiating to complex powers (Rogawski 611). Evaluated at z = π, this formula turns into the following famous identity:

$$e^{\pi i} + 1 = 0$$

Check Your Learning

1. Prove that $\frac{d}{dx} sin(x) = cos(x)$ using Taylor series.

2. Find the Maclaurin series for ln(1 - x^2) and state where it is valid (Rogawski 613).

Things you may Struggle With

1. Integrating/Differentiating series- It is often necessary to integrate or differentiate an infinite *Taylor series* term-by-term to arrive at a solution. Of course, we do not actually integrate every term in the series, since that would take forever. Instead, we integrate the form of every term in the series, which will include n as an index. n looks like a variable, but it is important that when integrating term-by-term, we treat the index n as a constant, using x as our variable of integration, just like we did in the sin(x^2) example.

2. Multiplying series- One can sometimes figure out what the product of two series will look like, just by multiplying the first few terms using FOIL. In this event, if one wants to get a glimpse at the terms of the product series up to, say degree 5, one should FOIL out the two series, throwing out terms that result in a power of x greater than 5. Combining like terms algebraically, the result is the beginning of the *Taylor series expansion* of the product.

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1.

$$\frac{d}{dx}sin(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{d}{dx}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = cos(x)$$

2.

$$ln(1 - x^2) = -\sum_{n=1}^{\infty}\frac{x^{2n}}{n} \text{ on the interval } (-1, 1)$$

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.