Week 3
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 3 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *(In)Definite Integral, Integrand, U-Substitution, Integration by Parts, Trig Identities*

Topic of the Week: Techniques of Integration

Contents:

Highlight 1: 5.7 Integration by Substitution

Consider the following indefinite integral: $\int(3x^2 + 4x + 3) * \cos(x^3 + 2x^2 + 3x + 7)\, dx$

This integral cannot be evaluated using simply the memorized power rule or trig rules. It requires some insight from the mathematician. The method of *integration by substitution* is to ask oneself the question: "Can the derivative of any one piece of the *integrand* be factored from the *integrand* elsewhere?" Note that the *integrand* in question consists of a couple pieces, notably a "$3x^2 + 4x + 3$" as well as a "$x^3 + 2x^2 + 3x + 7$" lodged within a cosine function. Notice that the former is simply the derivative of the latter. This means we can use the *U-substitution* method. The algorithm runs as follows.

Let $u = x^3 + 2x^2 + 3x + 7$. This is one of the terms we will be *substituting*.

Therefore, $\frac{du}{dx} = 3x^2 + 4x + 3$.
Rearranging, $dx = \frac{du}{3x^2+4x+3}$. This is the other term we will be *substituting*.
Rewrite the original $\int (3x^2 + 4x + 3) * \cos(x^3 + 2x^2 + 3x + 7)\, dx$ with the *substitutions* like this: $\int (3x^2 + 4x + 3) * \cos(\mathrm{u}) \frac{du}{3x^2+4x+3}$.
Notice that there is a $3x^2 + 4x + 3$ in both the numerator and denominator of the *integrand*.
Cancelling, we now have $\int \cos(\mathrm{u})\, du$.
From here, we apply the memorized rule for the integration of the cosine function to get $\int \cos(\mathrm{u})\, du = \sin(u) + C$.
To complete the problem, put everything back in terms of the original variable.
Therefore,
$\int (3x^2 + 4x + 3) * \cos(x^3 + 2x^2 + 3x + 7)\, dx = \sin(x^3 + 2x^2 + 3x + 7) + C$.
If we check this result by differentiating using the chain rule, we find that indeed, the equality holds true.

The method of *U-substitution* is the same for definite integrals, but one must remember to switch the bounds of integration while integrating with respect to u, rather than x.

Highlight 2: 7.1 Integration by Parts

The product rule states that $\frac{d}{dx}(u * v) = u * \frac{dv}{dx} + v * \frac{du}{dx}$. Integrating both sides with respect to x, $\int \frac{d}{dx}(u * v)dx = \int (u * \frac{dv}{dx} + v * \frac{du}{dx})$ dx.
Simplifying, $uv = \int u * \frac{dv}{dx} dx + \int v * \frac{du}{dx} dx = \int udv + \int vdu$ or, most usefully,

$$\int udv = uv - \int vdu.$$

This is the *integration by parts* formula.

When the *integrand* is a product, and *U-substitution* doesn't work, *integration by parts* may be the only way to evaluate the integral analytically.

For example, consider the integral $\int xe^x dx$.
The first step of the algorithm is to choose a u and a dv. We often want to choose a u that will get simpler upon differentiation, although this is not always the case. We let $u = x$ and $dv = e^x dx$, which is the only thing remaining. The next step of the algorithm is to find du and v.
$\frac{du}{dx} = 1$, so $du = dx$, and $v = \int dv = \int e^x dx = e^x + D$. As it turns out, when doing *integration by parts*, we can ignore this last constant D, as it will always end up cancelling.
Now, applying the formula,
$\int xe^x dx = xe^x - \int e^x dx = e^x(x - 1) + C$.

It is not always possible to get the integral to a recognizable form. However, in some cases, one can *integrate by parts* repeatedly, and it may be possible to solve for the original integral algebraically.

Highlight 3: 7.2 Trigonometric Integration

Basic form: $\int sin^m x * cos^n xdx$. The idea is to apply a trig identity to manipulate the integrand to a point where you can use *U-substitution* or *integration by parts*.

$$\int \sin^3 xdx$$

Consider the above integral.
Notice that we can separate the *integrand* into two terms: a sine and a sine squared. This enables us to apply a useful *trigonometric identity*.

$$\int (1-\cos^2 x)(\sin x)dx = \int \sin xdx - \int \sin x \cos xdx = \frac{1}{2}\cos^2 x + c$$

We see that the trig identity allows us to use *integration by parts* to get the integrand to a form we recognize. From there, it is simply a matter of applying the memorized formula for integrating sine*cosine.

Consider another example:

$$\int \sin^4 x \cos^5 xdx$$

Applying the appropriate *trig identity*, we obtain the following:

$$\int (\sin^4 x)(\cos x)(1-\sin^2 x)^2 dx = \int (\sin^4 x - \sin^6 x + \sin^8 x)\cos xdx$$

Now using u-substitution we can set $u = \sin x$, so then $du = \cos x$, therefore we have

$$\int (u^4 + u^8 - u^6)du = \frac{u^5}{5} + \frac{u^9}{9} - \frac{u^7}{7} + C$$

$$= \frac{sin^5 x}{5} + \frac{\sin^9 x}{9} - \frac{\sin^7 x}{7} + C$$

Highlight 4: 7.3 Trigonometric Substitution

Oddly enough, trigonometry is used to integrate integrands that do not contain trig functions at all. There are a few special forms an integrand can take where the most efficient way to solve the problem is to apply a *trigonometric substitution*. These forms are as follows:

$$\frac{1}{\sqrt{a^2-x^2}}, \frac{1}{x^2\sqrt{x^2-a^2}}, \frac{1}{\sqrt{x^2+a^2}},$$

where "a" is a constant and "x" is the variable of integration. The best way to understand how this is possible is to look at an example.

$$\int \frac{1}{\sqrt{1-x^2}}dx$$

Letting $x = a * \sin(\theta)$ such that $dx = \cos(\theta)d\theta$,

$$\int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta \tag{19}$$

if we look closely we see that the $1-\sin^2\theta$ is a trig-identity that is equal to $\cos^2\theta$. Therefore we can rewrite the integral

$$\int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}} = \int \frac{\cos\theta d\theta}{\cos\theta} = \int d\theta \tag{20}$$

Therefore the solution is $\theta + c$, but since we need the answer in terms of x we must solve $x = \sin\theta$ for θ. Solving for θ yields $\theta = \arcsin x$, which means our final solution is:

$$\boxed{\frac{1}{\sqrt{1-x^2}} dx = \arccos x + c = \cos^{-1} + c} \tag{21}$$

Consider an example for the second form.

$$\frac{1}{x^2\sqrt{x^2-9}}$$

since we see that $a = 3$ we know that we can use $x = 3\sec\theta$ which allows to also use $dx = 3\sec\theta\tan\theta$ rewrite the integral as follows;

$$\int \frac{3\sec\theta\tan\theta d\theta}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2-3^2}} = \int \frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}} \tag{23}$$

Now we can apply the trig-identity $\sec^2\theta - 1 = \tan^2\theta$ to rewrite the integral again as:

$$\int \frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta\sqrt{9\tan^2\theta}} = \int \frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta 3\tan\theta} = \int \frac{d\theta}{9\sec\theta} \tag{24}$$

but since we know that $\frac{1}{\sec\theta} = \cos\theta$ it follows that we can rewrite the integral again:

$$\int \frac{d\theta}{9\sec\theta} = \frac{1}{9}\int \cos\theta d\theta = \frac{1}{9}\sin\theta + c \tag{25}$$

Since $x = 3\sec\theta \implies \sec\theta = \frac{x}{3}$. Notice that $\sec x = \frac{1}{\cos x} = \frac{x}{3}$. Thus we can say that $\cos x = \frac{3}{x} = \frac{\text{adjacent}}{\textit{hypotenuse}}$. Therefore, we can now set up a triangle where the hypotenuse is equal to x, the adjacent side is equal to 3 which means our opposite is equal to $\sqrt{x^2-9}$. Since $\sin\theta = \frac{\text{opposite}}{\textit{hypotenuse}} = \frac{\sqrt{x^2-9}}{x}$ it follows that $\frac{1}{9}\sin\theta = \frac{\sqrt{x^2-9}}{9x}$. Therefore the final answer to our initial integral is the following:

$$\boxed{\frac{1}{9}\sin\theta + c = \frac{\sqrt{x^2-9}}{9x} + c}$$

Consider an example of the final form.

$$\int \frac{1}{\sqrt{x^2+9}} dx \tag{27}$$

we see that if we let $x = 3\tan\theta$ then we can rewrite the integral as:

$$\int \frac{3\sec^2\theta d\theta}{\sqrt{(3\tan\theta)^2+3^2}} = \int \frac{3\sec^2\theta d\theta}{\sqrt{9\tan\theta^2+9}} \tag{28}$$

using trig-identities we know that $9\tan^2\theta + 9 = 9\sec^2\theta$ therefore we have:

$$\int \frac{3\sec^2\theta d\theta}{\sqrt{9\tan\theta^2+9}} = \int \frac{3\sec^2\theta d\theta}{\sqrt{9\sec^2\theta}} = \int \frac{3\sec^2\theta d\theta}{3\sec\theta} = \int 3\sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c \tag{29}$$

Once again we need our answer in terms of x instead of θ. Since $x = 3\tan\theta$ it follows that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3}$ and furthermore we know that the hypotenuse is equal to $\sqrt{x^2+9}$. Therefore we can use this triangle to help us change variables. Since $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$ we know that $\sec\theta = \frac{\sqrt{x^2+9}}{3}$. Which gives us our final solution:

$$\boxed{\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C} \tag{30}$$

The intuition behind all of these substitutions is the simple *identity* $sin^2\theta + cos^2\theta = 1$. In this *identity* is contained the idea of an additive/subtractive relationship between the square of a trig function, a constant, and the square of another trig function. A similar logic applies for the related identity $sec^2\theta = 1 + tan^2\theta$. The special forms of the above examples take advantage of these relationships to manipulate the *integrand*, even though the final solution may not contain trigonometric functions at all.

Most *integrands* are not integrable analytically. But problems like these test your ability to manipulate mathematical symbols.

Check Your Learning

1. $\int e^{-x} sin(x) dx$

(Rogawski 405)

2. $\int tan^5 x\, sec^4 x\, dx$

(Rogawski 413)

3. $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$

(Rogawski 420)

Things you may Struggle With

1. *Integration by parts*- Recall the formula for integration by parts is $\int udv = uv - \int vdu$. Note that the second integral is subtracted from the product uv. Make sure you keep your signs straight; it is very easy to accidentally drop a negative and get the wrong answer, even if you integrate correctly, especially if you have to use integration by parts multiple times.

2. *Trig Substitution*- It is sometimes confusing to know which Pythagorean trig identity you will need to use for integrating one of the special radical expressions. A helpful way to remember is that $cos^2x = 1 - sin^2x$, whereas $sec^2x = tan^2x + 1$. Notice how the constant is positive in the first case, while the trig function to be substituted is negative. And in the second case, both the constant and the trig function to be substituted are positive, or, with a rearrangement, the trig function is positive and the constant is negative. Keeping this in mind, pay attention to the signs of the variable and the constant under the radical, and choose the trig identity that matches those signs.

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. $-\frac{1}{2}e^{-x}(sin(x) + cos(x)) + C$

2. $\frac{1}{8}sec^8x - \frac{1}{3}sec^6x + \frac{1}{4}sec^4x + C$

3. $\frac{9}{2}sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.

Trigonometric integration and trigonometric substitution examples are taken from Ethan Reyes' tutoring resources.