Week 4
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is Week 4 of class, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Partial Fraction Decomposition*

Topic of the Week: 7.5 Method of Partial Fractions

Contents:

Highlight: 7.5 Partial Fractions Decomposition

It may be helpful to see the big picture of the method of *partial fractions*. Integrations that require *partial fraction decompositions* are approached differently depending on the form of the denominator of the integrand, namely, how the denominator factors. There are basically 5 forms of the denominator you may be responsible for recognizing. They are as follows:

1. non-repeated linear factors
2. repeated linear factors
3. irreducible quadratic factors
4. reducible quadratic factors
5. repeated quadratic factors

Add to these the method of simplifying using long division, and you will have everything you need to solve integrations involving *partial fractions decompositions*. Let us look at each case individually.

1. Non-repeated linear factors

This case is the most straightforward. Without loss of generality, here is the form of this *partial fraction decomposition*:

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

From this point on, all *partial fraction decompositions* are calculated the same. The method is to multiply both sides of the equations by the common denominator and then group everything on each side by powers of x (or whatever the variable of integration may be). Then, equate the coefficients of each power of x — one side of the equation to the other. This will give a system of linear equations that one can solve using methods learned in Algebra 2 or earlier. The goal is to calculate the unknown values A and B, so that one can complete the *partial fraction decomposition*. I will illustrate for this example, but all other examples follow in an almost identical fashion.

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

$$1 = A(x-5) + B(x-2)$$

$$1 = Ax - 5A + Bx - 2B$$

$$0x + 1 = (A+B)x + (-5A - 2B)$$

Thus, it must be true that $A + B = 0$ and that $-5A - 2B = 1$. Solving these equations for A and B, one obtains $A = -\frac{1}{3}$ and $B = \frac{1}{3}$. Substituting these values back into the original equation, one has completed the *partial fraction decomposition* and can now integrate (Rogawski 427).

It is often possible to avoid solving the resulting system of linear equations directly by picking clever values for x. For example, if, at the second step, we had let x=2, the B term would have gone to 0, and we would have been left with one equation and one unknown and could have quickly found the value of A. Then we could have found the value of B in a similar manner. This technique is preferable to solving the system by brute force, because it can save a great deal of time for more complicated problems. But if nothing else readily presents itself, it is always possible to solve for the constants as one would any linear system.

2. Repeated linear factors

Without loss of generality, the form of this *partial fraction decomposition* is as follows:

$$\frac{3x-9}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}$$

Here, the method is almost the same as for non-repeated linear factors, except that for the repeated factor, you will need 2 terms — one with a constant over the repeated factor and one with a constant over the repeated factor squared. If a factor is repeated 3 times (cubed), you will need to add an additional term to the *decomposition*: a constant over the repeated factor cubed, and the pattern continues. All other linear factors in the denominator are treated as they would be in the non-repeated case (Rogawski 428).

3. Irreducible quadratic factors

Without loss of generality, this *partial fraction decomposition* proceeds as follows:

$$\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9}$$

Notice that because the quadratic factor is not in turn factorable into linear elements, we must include a linear term in the numerator of the respective term in the decomposition — that is, the numerator is one degree less than the denominator. In fact, a similar pattern applies for irreducible factors of higher powers, although you shouldn't see any problems like that (Rogawski 430).

4. Reducible quadratic factor

Without loss of generality, this *partial fraction decomposition* proceeds in one of two ways:

$$\frac{18}{(x+3)(x^2-9)} = \frac{A}{x-3} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

OR

$$\frac{18}{(x+4)(x^2-9)} = \frac{A}{x+4} + \frac{B}{x-3} + \frac{C}{x+3}$$

Note that, depending on what other factors are in the denominator, this kind of *partial fractions decomposition* will resolve into either the repeated linear factor case or the non-repeated linear factor case. From there, it can be solved as would the above-mentioned forms (Rogawski 430).

5. Repeated quadratic factor

Without loss of generality, the form of this *partial fraction decomposition* is as follows:

$$\frac{4-x}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

One can see how this *decomposition* is, in effect, a hybrid of the irreducible quadratic factor case and the repeated linear factor case (Rogawski 431).

In all the above *partial fraction decompositions*, the numerator has not counted for anything. This is because the degree of the numerator has been strictly less than the degree of the denominator. This is a necessary condition for doing *partial fraction decompositions*. If the degree of the numerator is greater than or equal to the degree of the denominator, it will be necessary to do long division first, before attempting any *partial fraction decomposition*. For example, $\frac{x^3+1}{x^2-4}$ cannot be broken down immediately into *partial fractions*. But, if we divide like so,

$$\frac{x^3+1}{x^2-4} = \begin{array}{r} x \\ (x^2-4)\,\overline{)\,x^3+1} \\ -(x^3-4x) \\ \hline 4x+1 \end{array}$$

$$= x + \frac{4x+1}{x^2-4}$$

Now one can integrate the lone x by itself and proceed to apply a *partial fractions decomposition* to the fraction term in order to integrate (Rogawski 429).

A general framework for approaching *partial fractions decompositions*:

- 1st, make sure the degree of the numerator is less than the degree of the denominator (if not, use long division to make it so).
- 2nd, factor the denominator as much as possible.

- 3rd, recognize the category into which the problem falls. For Calculus 2 purposes, this will always be one of the above 5 categories.
- 4th, apply the form of the *partial fractions decomposition* and solve the resultant system of linear equations for the constants.

Check Your Learning

1. $\int \frac{dx}{x(2x+1)}$

2. $\int \frac{8\,dx}{x(x+2)^3}$

3. $\int \frac{25\,dx}{x(x^2+2x+5)^2}$

Things you may Struggle With

1. *Partial Fractions* is not integration. It is a method for simplifying an integrand to a point where it can be integrated. Once you perform a *partial fractions decomposition*, you must still integrate each of the resultant *fractions* individually in order to complete the original problem.

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. $ln|x| - ln|2x+1| + C$

2. $ln|x| - ln|x+2| + \frac{2}{x+2} + \frac{2}{(x+2)^2} + C$

3. $ln|x| - \frac{1}{2}|x^2+2x+5| + \frac{15-5x}{8(x^2+2x+5)} - \frac{13}{16}tan^{-1}\left(\frac{x+1}{2}\right) + C$

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.