Week 5
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is <u>Week 5 of class</u>, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *L'Hopital's Rule, Improper Integral, Limit, Convergence, Divergence*

Topic of the Week: More Techniques of Integration

Contents:

Highlight 1: 4.5 L'Hopital's Rule

L'Hopital's Rule is a formula that provides a method for solving *limits* of an indeterminate form. The indeterminate forms are as follows:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 * \infty, \frac{1}{0 * \infty}$$

To recognize the above forms, replace x as it appears in the *limit* expression for the value x is said to be approaching. If, when simplified, the *limit* expression has one of the above forms, *L'Hopital's Rule* is appropriate.

L'Hopital's Rule states that for *limits* of the indeterminate forms,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In English, if your *limit* is an indeterminate form, the *limit* of the expression is equal to the *limit* of the derivative of the numerator over the derivative of the denominator (Rogawski 224). In practicxe, this means you can simply find the derivative of the numerator, find the derivative of the denominator, and then take the *limit* of the resultant quotient. If the resultant quotient is still indeterminate, you can apply *L'Hopital's Rule* again and again until the quotient is determinate. This is why students tend to love *L'Hopital's Rule* so much; it is quick and easy and gets you to the right answer.

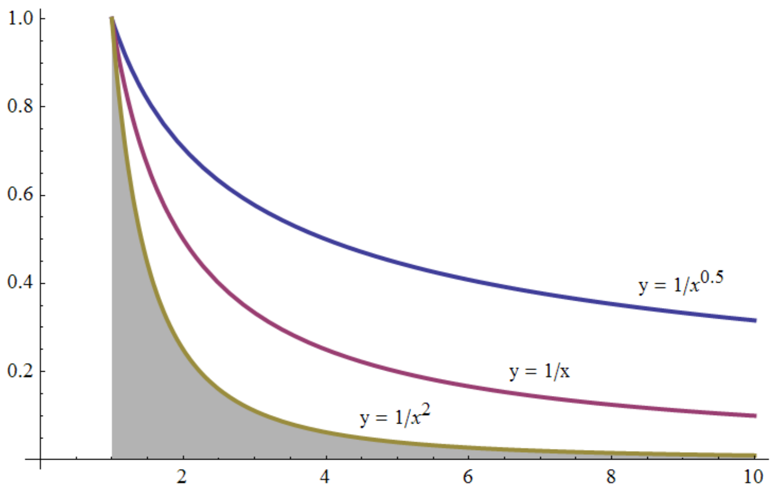
But remember not to get carried away; it cannot be emphasized enough that *L'Hopital's Rule* only applies to *limits* of an indeterminate form. You will be tempted to apply *L'Hopital's Rule* to any tricky-looking *limit* expression. But you must resist this temptation. If you try to apply *L'Hopital's Rule* to *limits* that are not in the indeterminate form, you are almost guaranteed to get the wrong answer.

Highlight 2: 7.7 Improper Integrals

What is one to make of the expression $\int_a^\infty f(x)dx$? As of yet, we have not actually defined what it means to integrate a function all the way to infinity. Technically speaking, writing the above expression is meaningless, a la the Fundamental Theorem of Calculus, because it would mean plugging the number ∞ into the antiderivative of the integrand. But ∞ is not a number that can be plugged into a function the way 5 or 8 can. But mathematicians have found it useful to talk about something sort of like an infinite integral. Therefore, to make sense of the above expression, we define the *improper integral* as follows:

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx$$

An identical logic applies for integrating from -∞ to "a." If the *limit* exists, the integral is said to *converge*; it *diverges* otherwise (Rogawski 442).



(oak.ucc.nau.edu)

This definition of the *improper integral* prompts a similar definition for integrating a function that approaches ±∞ as x approaches some finite value b:

$$\int_a^b f(x)dx = \lim_{R\to b^-} \int_a^R f(x)dx$$

A similar logic applies if the function goes infinite at the lower bound. In this case, you will take the *limit* as x approaches "a" from the right. Again, the integral is said to *converge* if the *limit* exists; it *diverges* otherwise (Rogawski 446).

The concept of an "*improper integral*" reduces the problem of "infinite integrals" into two simpler problems: finding the antiderivative of the integrand and finding a *limit* of the antiderivative.

Some interesting mathematical results flow from these definitions of the *improper integral*, such as the *divergence* of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the paradoxical properties of Gabriel's Horn. There are also numerous applications (illustrated in the textbook) to fields as diverse as physics and finance. Formulas can be derived to provide rules for integrating special infinite forms like $\frac{1}{x^p}$. Without further discussion, the rules for integrating integrands of this form are as follows:

$$\int_a^\infty \frac{dx}{x^p} = \left\{ \begin{matrix} \frac{a^{1-p}}{p-1} & if\ p > 1 \\ diverges & if\ p \leq 1 \end{matrix} \right\}$$

$$\int_0^a \frac{dx}{x^p} = \left\{ \begin{matrix} \frac{a^{1-p}}{p-1} & if\ p < 1 \\ diverges & if\ p \geq 1 \end{matrix} \right\}$$

These rules for determining *convergence*/*divergence* of the special form $\frac{1}{x^p}$ will primarily be of use when applying the comparison test (Rogawski 443, 447).

The comparison test is a method of proving the *convergence* or *divergence* of one integral by proving the *convergence* or *divergence* of another integral. The formulae for the comparison test are as follows: If f and g are continuous, and $f(x) \geq g(x) \geq 0$, then

the *convergence* of $\int_a^\infty f(x)dx$ implies the *convergence* of $\int_a^\infty g(x)dx$ and
the *divergence* of $\int_a^\infty g(x)dx$ implies the *divergence* of $\int_a^\infty f(x)dx$.

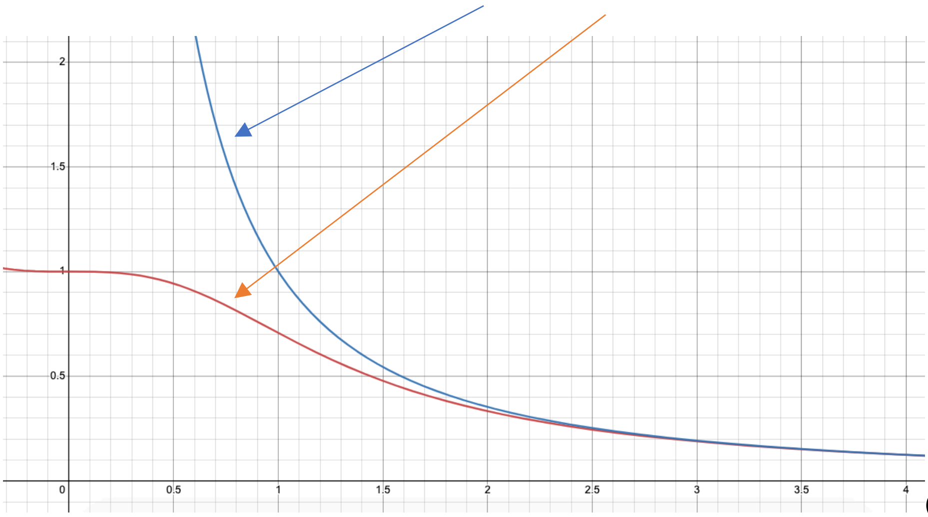
The intuition behind this is straightforward. If one area is smaller than another finite area, then the first area must also be finite. If one area is larger than another infinite area, then the first area must also be infinite.

For example, suppose we are asked to show that the following form *converges*:

$$\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$$

Because we are asked to prove *convergence*, we are in search of a function that *converges* when integrated and is always larger than the above integrand. Observe:

$$x^3 \leq x^3 + 1$$
$$\sqrt{x^3} \leq \sqrt{x^3 + 1}$$
$$\frac{1}{x^{3/2}} \geq \frac{1}{\sqrt{x^3 + 1}}$$



(desmos.com)

Here, we have found a function that is greater than our integrand on the range we wish to integrate. Furthermore, by the $\frac{1}{x^p}$ rule, we know that $\int_1^\infty \frac{dx}{x^{3/2}}$ *converges* to $\frac{1^{1-3/2}}{3/2-1}$. This exact value is not important; all that matters is that we have found a function that is larger than our original integrand and that when integrated over the same range, *converges*. Thus, by the comparison test for *improper integrals*, we can say with certainty that $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ *converges* (Rogawski 448).

Check Your Learning

1. $\lim_{x\to 0} \frac{\sin(x)}{x}$

2. Does the following integral *converge* or *diverge*: $\int_0^\infty \frac{dx}{1+x}$

Things you may Struggle With

1. *Partial Fractions* is not integration. It is a method for simplifying an integrand to a point where it can be integrated. Once you perform a *partial fractions decomposition*, you must still integrate each of the resultant *fractions* individually in order to complete the original problem.

2. *L'Hoptial's Rule* only applies for limit expressions of the indeterminate form. If the limit expression is not in an indeterminate form, *L'Hopital's Rule* does not apply.

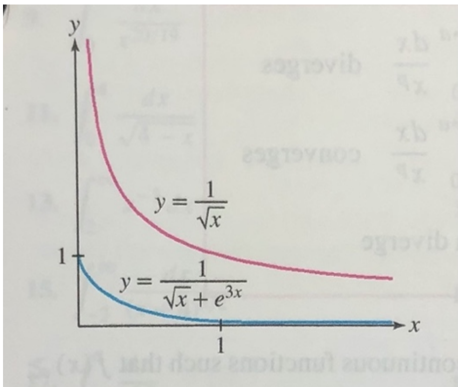


FIGURE 11 The divergence of the integral of the larger function says nothing about the integral of the smaller function.

3. When asked to determine whether an *improper integral converges* or *diverges*, it will often be necessary to try to prove both — to fail at proving one before eventually proving the other. This is not your fault. Just try finding a function (often in the form $\frac{1}{x^p}$) that is larger than the integrand and *convergent* when integrated, or smaller than the integrand and *divergent* when integrated. The order of this is important. If you mismatch these, nothing is proved; you would be simply showing, for instance, that a function which is larger than your integrand *diverges* when integrated, which doesn't let you infer anything about the desired integrand.

(Rogawski 449)

Answers to Check Your Learning

1. 1

2. *diverges*

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.