Week 8
MTH-1322 – Calculus 2

Hello and Welcome to the weekly resources for MTH-1322 – Calculus 2!

This week is Week 8 of class, and typically in this week of the semester, your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to **look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester**.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135.

Keywords: *Force, Mass, Acceleration, Work, Energy, Arc Length, Surface Area*

Topic of the Week: Work, Arc Length, & Surface Area

Contents:

Highlight 1: 6.5 Work and Energy

Integrals have many applications in physics. One important application is *work*. *Work* is a measure of the amount of *energy* expended. *Energy* is expended when a *mass* is *accelerated* from one location in space to another. Oftentimes, this *acceleration* takes the form of a *force* needed to overcome a *force* of resistance, such as gravity or kinetic friction. By Newton's second law, *mass* times *acceleration* is *force*. Hence, *work* may be thought of as the product of *force* and distance. When *force* is variable across distance, to get *work*, one must integrate *force* with respect to distance.

$$W = \int_a^b F(x)dx$$

Work is given in units of Joules, which are reducible to $\frac{kg*m^2}{s^2}$. In this definition, we can see the irreducible SI units of measurement, indicating the fundamentally derivative nature of *energy*. We observe also that since *force* is *mass* (kg) times *acceleration* (m/s^2), and distance is just meters (m), it is only natural that *work* would be *force* times distance or

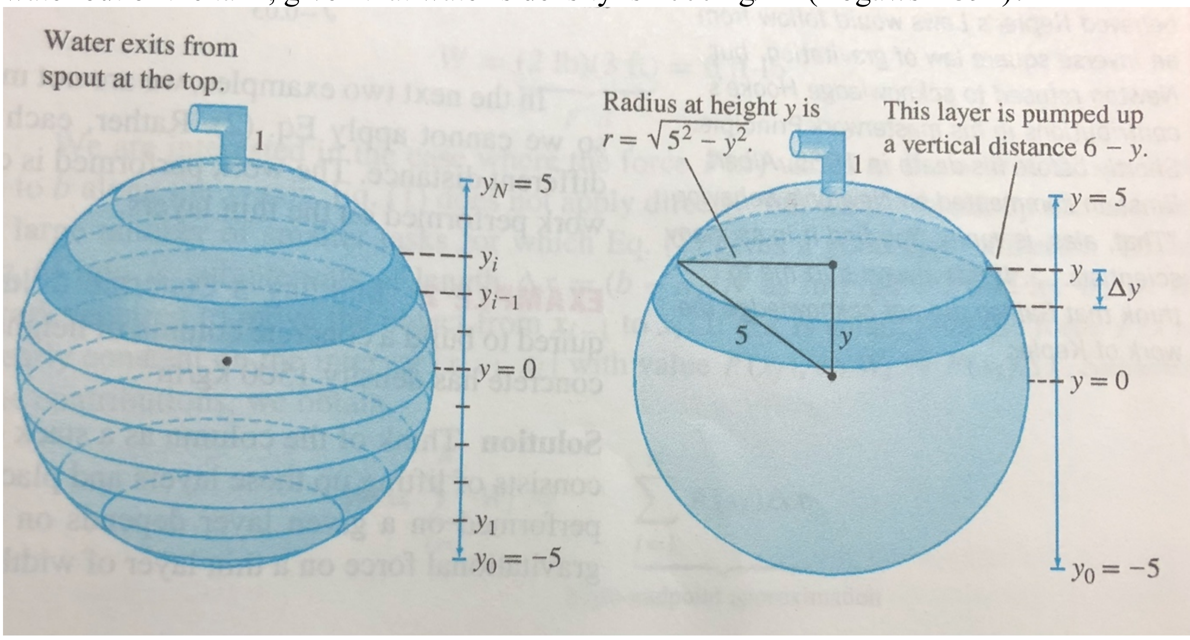
$$Work = force * distance = mass * acceleration * distance \ = kg * \frac{m}{s^2} * m = \frac{kg * m^2}{s^2}$$

For example, consider a problem asking us to find the *work* done on a spring 10 cm beyond its resting place, if it has a spring constant of 400 N/m. Hooke's Law states that a spring exerts a *force* equal to its spring constant times the spring's displacement. Therefore, we know that the *force* function looks like this: $F(x) = 400x$. All that remains to find the *work* done on the spring is to integrate this function with respect to distance over the length of the displacement. Converting centimeters to meters, to match the units of the *force* function,

$$\int_0^{0.1} 400x dx = 200 * 0.1^2 - 200 * 0^2 = 2\,J$$

And our result is in units of Joules (Rogawski 393).

Consider another example, one where we are asked to find the *work* done to pump water out of a tank. Let us say we have a spherical tank of radius of 5 meters that is full of water. If water leaves the tank through a 1-meter spout on top, how much *work* is done to pump all the water out of the tank, given that water's density is 1000 kg/m^3 (Rogawski 394)?



The trick is to think of the sphere as divided into pancake-like disks, where the width of the disk is Δy. The game plan is to first figure out the *work* done to raise one of these disks up through the spout and then integrate at the end over the whole desired range to get total *work*.

We express the approximate area of the top of the disk as a function of height y as follows: $A(y) = \pi(25 - y^2)$. Multiplying area of the top of the disk by the disk width yields volume: $V(y) = A(y)\Delta y$. Multiplying volume by density yields *mass*: $M(y) = 1000 * V(y)$. Multiplying mass by gravitational *acceleration* yields force: $F(y) = M(y) * 9.8$. Tying all of this together, we have our *force* function as follows: $F(y) = 9.8 * 1000\pi(25 - y^2)\Delta y$.

The *work* done to lift this layer of water out the top spout is the *work* done against the *force* of gravity over the distance from the layer to the top of the spout. Consistent with our coordinate plane, this distance may be represented as (6 – y) where y is the position on the vertical axis illustrated above. We can find the *work* done on this one layer of water by simply multiplying our *force* function by this distance function, evaluated at the appropriate y. But since we desire to find the *work* done when pumping all of the water out of the tank, we must integrate *force* times distance over the entire relevant y axis. Thus, the problem becomes

$$\int_{-5}^{5} 9.8 * 1000\pi(25 - y^2)(6 - y)dy = 31{,}000{,}000\, J$$

Highlight 2: 8.2 Arc Length & Surface Area

It may or may not come as a surprise to you that *arc length* can be interpreted and calculated as an integral. The intuition behind this principle is that finding *arc length* amounts to the problem of finding the sum of the lengths of the hypotenuses of ever smaller triangles, of which the base is Δx and the height is $f(x + \Delta x) - f(x)$ (see Rogawski 437 for more details). One can begin to see the limit definition form of the derivative in this intuition, so one might suspect that the formula for *arc length* will involve derivatives. One would be correct.

$$s = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

A single example should suffice to illustrate the application of this formula.

EXAMPLE 1 Find the arc length s of the graph of $f(x) = \frac{1}{12}x^3 + x^{-1}$ over the interval $[1, 3]$ (Figure 4).

Solution First, let's calculate $1 + f'(x)^2$. Since $f'(x) = \frac{1}{4}x^2 - x^{-2}$,

$$1 + f'(x)^2 = 1 + \left(\frac{1}{4}x^2 - x^{-2}\right)^2 = 1 + \left(\frac{1}{16}x^4 - \frac{1}{2} + x^{-4}\right)$$

$$= \frac{1}{16}x^4 + \frac{1}{2} + x^{-4} = \left(\frac{1}{4}x^2 + x^{-2}\right)^2$$

Fortunately, since this expression for $1 + f'(x)^2$ is a square, the arc-length integral simplifies nicely and is easily computed:

$$s = \int_1^3 \sqrt{1 + f'(x)^2}\,dx = \int_1^3 \left(\frac{1}{4}x^2 + x^{-2}\right) dx = \left(\frac{1}{12}x^3 - x^{-1}\right)\Bigg|_1^3$$

$$= \left(\frac{9}{4} - \frac{1}{3}\right) - \left(\frac{1}{12} - 1\right) = \frac{17}{6}$$

(Rogawski 474)

A handy formula also exists to calculate the *surface area* of a volume of rotation after the manner of the disk and shell methods chapters. Like the disk and shell methods, this formula is useful when dealing with a solid obtained from rotating a known 2-dimensional function about an axis in three-space. Unlike the disk and shell method, this formula is used to calculate *surface area*, not volume.

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}dx$$

Note the similarity to the *arc length* formula. The intuition behind this similarity is that one is essentially doing the same calculation, except that at each x value along the function, one is finding the circumference of a circle and multiplying that by a small change in x to get units of area.

Consider the following example.

EXAMPLE 5 Find the surface area of the surface, called a paraboloid, that is obtained by rotating the graph of $f(x) = \sqrt{x}$ about the x-axis for $0 \le x \le 1$.

Solution The graph of $f(x) = \sqrt{x}$ is the top half of a parabola opening along the x-axis, which becomes a paraboloid when rotated about the x-axis (Figure 11). Then $f'(x) = \frac{1}{2\sqrt{x}}$ and hence, we obtain

$$S = 2\pi \int_0^1 f(x)\sqrt{1 + f'(x)^2}\,dx = 2\pi \int_0^1 \sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\,dx$$

$$= 2\pi \int_0^1 \frac{\sqrt{x}}{2\sqrt{x}}\sqrt{4x + 1}\,dx = \pi \int_0^1 \sqrt{4x + 1}\,dx$$

$$= \frac{\pi}{6}(4x + 1)^{3/2}\Big|_0^1 = \frac{\pi}{6}(5^{3/2} - 1) \approx 5.3304$$

(Rogawski 477)

Check Your Learning

1. Find the *work* done against gravity to construct the Great Pyramid of Giza, which is 146 meters high with a square base of side 230 meters (density of rock is 2000 kg/m^3).

2. Find the *length of the arc* of the curve $x^2 = (y - 2)^3$ from (1, 3) to (8, 6).

Things you may Struggle With

1. *Work/energy* problems can be difficult to conceptualize. If you are not given a picture of the scenario, it is always a good idea to draw your own. Be sure you label the axis of integration, as this will help you set up the ultimate *work* integral. Then, try to express *force*

as a function of the variable of integration. As seen in our example, getting to this state might involve passing through a number of intermediate stages (area, volume, density, *mass*).

2. Due to the complexity added to an integral by the *arc length* formula, it is not unusual that the result is not integrable analytically. This can make *arc length* and *surface area* problems seem intimidating. For calc 2 purposes though, problems are designed so that they can be evaluated analytically. Look for little tricks, like completing the square in our example, that will simplify the ultimate integral — such convenient tricks will often exist in contrived problems and will enable you to solve *arc length* problems that look hopeless at first.

Thanks for checking out these weekly resources!
Don't forget to check out our website for group tutoring times, video tutorials and lots of other resources: www.baylor.edu/tutoring ! Answers to check your learning questions are below!

Answers to Check Your Learning

1. 1,842,000,000,000 Joules

2. 7.6337

References

Rogawski, Jon, et al. *Calculus: Early Transcendentals*. W.H. Freeman, Macmillan Learning, 2019.