Week 10
MTH-2321 - Calculus III

Hello and Welcome to the weekly resources for MTH-2321 - Calculus III!

This week is Week 10 of class, and typically in this week of the semester. your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135

Key Words: Scalar Line Integrals, Vector Line Integrals, Conservative Vector Fields

TOPIC OF THE WEEK

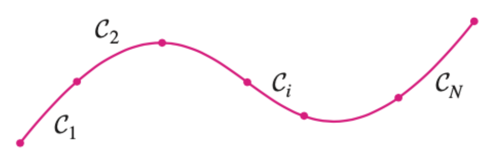
Line Integrals

Line Integrals: integrals over a curve

Two types: integrals of functions (scalar) and integrals of vector fields (vector)

Scalar Line Integrals

- Function f over a curve
- Represent total mass and charge
- Summation of small curves to make up the full curve



Partition of $\mathcal{C}$ into N small arcs

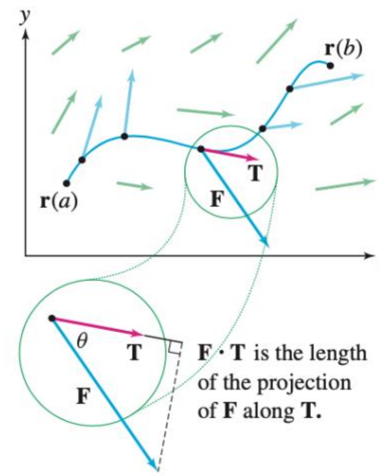
Let $\boldsymbol{r}(t)$ be a parameterization of a curve C for $a \le t \le b$.

$$\int_C f(x,y,z)\,ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\,dt$$

Recall that $\|r'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

Also, the arc length, s is often referred to as the <u>line element</u> or <u>arc length differential</u>. Therefore,

$$ds = \|r'(t)\|\ dt$$



Vector Line Integrals: the line integral of a vector field **F** along an oriented curve C is the tangential component of **F**

- Main difference between scalar and vector: vector line integrals <u>depend on the direction of the curve</u> (called the orientation)

$$\int_C \boldsymbol{F}\ d\boldsymbol{r} = \int_C (\boldsymbol{F} \cdot \boldsymbol{T})\ ds$$

At a point P, recall that, $\boldsymbol{F}(P) \cdot \boldsymbol{T}(P) = \|\boldsymbol{F}(P)\|\|\boldsymbol{T}(P)\| \cos\theta = \|\boldsymbol{F}(P)\| \cos\theta$

- T is a unit tangent vector and therefore has a magnitude of 1

We typically use parameterization, which allows the equation to then look like this:

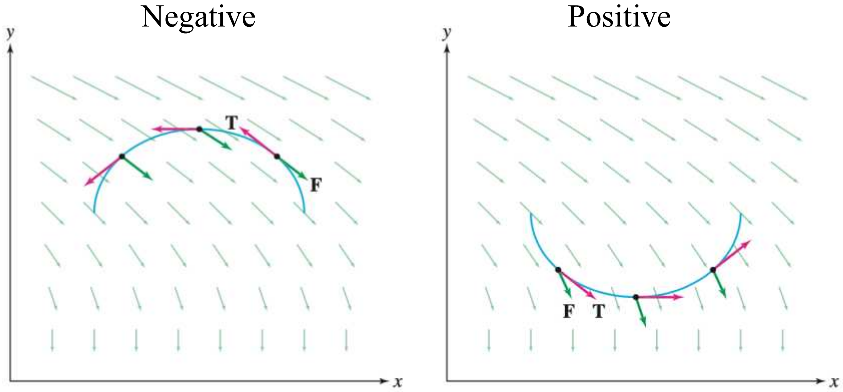
$$\int_C \boldsymbol{F}\ d\boldsymbol{r} = \int_a^b \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\ dt$$

It can also be broken up by variable as such:

$$\int_C F_1\ dx + F_2\ dx + F_3\ dx = \int_a^b \left(F_1\big(\boldsymbol{r}(t)\big)\frac{dx}{dt} + F_2\big(\boldsymbol{r}(t)\big)\frac{dy}{dt} + F_3\big(\boldsymbol{r}(t)\big)\frac{dz}{dt}\right)\ dt$$

The sign of the vector line integral is based on the angle between F and T. If the angle is obtuse ($> 90°$), then the line integral is negative and then we are working against the vector field. If the angle is acute ($< 90°$), then the line integral is positive, and we are working with the vector field.

Negative Positive



HIGHLIGHT #1: Applications of Line Integrals

Scalar Line Integrals: if we view curve C as a wire with a continuous mass density $\delta(x, y, z)$, then the total mass of the wire is defined as the integral of mass density.

- This could also be done with charge density and total charge density!

$$Total\ mass\ of\ C = \int_C \delta(x, y, z)\, ds$$

- They can also be used to calculate electric potentials (aka voltages, V) which is caused by a continuous charge density along a curve.
  - The variable k is a constant where $k = 8.99 \times 10^9 \frac{N-m^2}{C^2}$
  - dp is a function denoting the distance from (x, y, z) to P

$$V(P) = k \int_C \frac{\delta(x, y, z)}{dp}\, ds$$

Vector Line Integrals: most commonly used to solve for work which refers to the energy expended when a force to an object over a distance

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

If asked to find the work required to move against the presence of a force field **F** then

$$W = -\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

HIGHLIGHT #2: Conservative Vector Fields

Conservative Vector Fields: A vector field, F is conservative if and only if $curl(F) = 0$ or

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

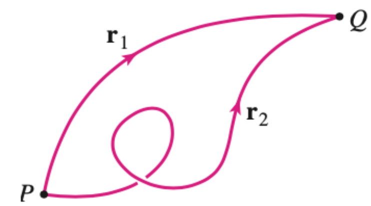
Also, the curve must by simply connected, meaning there are no holes in the domain D.

If the curve is closed, we refer to the line integral as the circulation of **F** around C and it is denoted as,

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

Conservative vector fields are path independent. This means that the line integral of **F** only depends on the end points and not the particular path followed.

- Therefore, in the figure to the right, the line integral over r_1 would be the same as over r_2.



$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(Q) - f(P) = 0$$

Potential Functions: a vector field is conservative if it has a potential function f so that $\nabla f = F$.

How to find the potential function:

Ex. Show that $\boldsymbol{F} = \langle 2xy + y^3, x^2 + 3xy^2 + 2y \rangle$ is conservative and find the potential function.

Step 1: observe the cross-partial derivatives to determine if **F** is conservative

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xy + y^3) \qquad = 2x + 3y^2$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy^2 + 2y) = 2x + 3y^2$$

They are equal!

Step 2: Is the domain simply connected?

- Yes, because **F** is not undefined anywhere along the path

Therefore, **F** is a conservative vector field!

Step 3: Find the potential function

$$\frac{\partial f}{\partial x} = F_1(x, y) = 2xy + y^3 \qquad and \qquad \frac{\partial f}{\partial y} = F_1(x, y) = \ x^2 + 3xy^2 + 2y$$

$$f(x,y) = \int F_1(x,y)$$

$$f(x,y) = \int (2xy + y^3)\,dx$$

$$f(x,y) = x^2y + xy^3 + g(y)$$

$$f(x,y) = \int F_2(x,y)$$

$$f(x,y) = \int (x^2 + 3xy^2 + 2y)\,dy$$

$$f(x,y) = x^2y + xy^3 + y^2 + h(x)$$

Must add arbitrary constant depending on x and y alone into both! It's like adding +C but more specific

Step 4: Compare the two expressions for $f(x,y)$ because they must be equal.

$$x^2y + xy^3 + g(y) = x^2y + xy^3 + y^2 + h(x)$$

Therefore, we see that $g(y) = y^2$ and $h(x) = 0$. So the general potential function of **F** is:

$$f(x,y) = x^2y + xy^3 + y^2 + C$$

Useful videos for further explanation of change of variable and vector fields!

Full Playlist about Line Integrals
https://www.youtube.com/playlist?list=PLX2gX-ftPVXVLU50P8azDmZZLRkfmBI1Q

Conservative Vector Fields
https://www.youtube.com/watch?v=76nzOtupeRc

CHECK YOUR LEARNING

1. Calculate $\int_C (x + y + z)\,ds$ where C is the helix $r(t) = (\cos t\,, \sin t\,, t)$ for $0 \le t \le \pi$.

2. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F} = \langle z, y^2, x \rangle$ and C is parameterized by $\boldsymbol{r}(t) = (t+1, e^t, t^2)$ for $0 \le t \le 2$.

3. Find the total mass of a wire in the shape of a parabola $y = x^2$ for $1 \le x \le 4$ (in centimeters) with mass density given by $\delta(x, y) = \frac{y}{x}$ g/cm.

4. Find a potential function for $\boldsymbol{F} = \langle 2xyz^{-1}, z + x^2z^{-1}, y - x^2yz^{-2} \rangle$.

THINGS YOU MAY STRUGGLE WITH

1. For line integrals just make sure you understand parameterization and how to integrate it into the problem. Most of the time you will be given the parameterization so just be sure you are using the right variables whether they are x, y, and z or t.

2. Make sure that you check if a vector field is conservative before assuming so. Make sure that the cross partials are equal or that the curl of the function is equal to zero. Also ensure the function is simply closed and there are no holes or undefined points in the domain.

That's all for this week! I hope this was a helpful review of 16.2 and 16.3! Feel free to visit the Tutoring Center Website for more information at www.baylor.edu/tutoring.

Answers:

1. $2\sqrt{2} + \frac{\sqrt{2}}{2}\pi^2$
2. $\frac{1}{3}(e^6 + 35)$
3. $Total\ mass \approx 42.74$
4. $f(x, y, z) = x^2yz^{-1} + yz + C$