Week 12
MTH-2321 - Calculus III

Hello and Welcome to the weekly resources for MTH-2321 - Calculus III!

This week is Week 12 of class, and typically in this week of the semester. your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135

Key Words: Green's Theorem, Boundary Curves

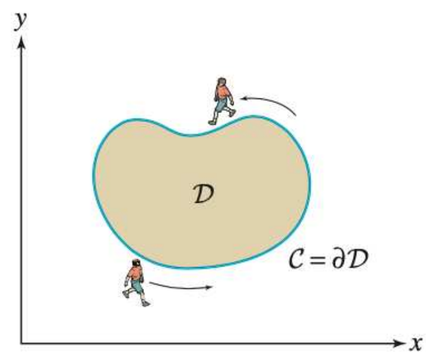
TOPIC OF THE WEEK

Green's Theorem

Recall that, a vector field **F** is conservative for a closed path when the $curl(\boldsymbol{F}) = 0$. So, what happens when **F** is not conservative? This is why we have Green's Theorem!

Notation and Vocabulary

- Simple closed curve: a closed curve, C that does not intersect itself
- The curve C bounds the domain D so the curve is denoted ∂D
- The boundary orientation of ∂D is the direction that if you were walking along the curve, the curve would always be on your left (as seen by the walking person in the picture to the right)
    - So typically, this is counterclockwise which we will call the positive orientation



Equation

Let D be a domain whose boundary ∂D is a simple closed curve, oriented clockwise,

$$\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

When can Green's Theorem be used?

1. It **can only** be used for simply closed curves
2. It **can only** be used for two-dimensional vector fields
3. It **can** be used to make calculations simpler for shapes that are more difficult to parameterize (ex. triangles, squares, etc.)

HIGHLIGHT #1: Computing Area and Additivity of Circulation

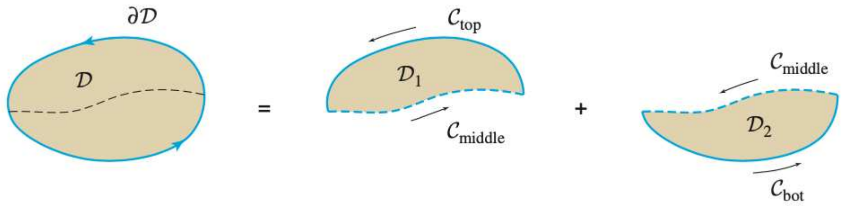
Computing Area: we can use Green's theorem to obtain formulars for the area of the domain D enclosed by the simply closed curve C

- To do this one must pick a vector field such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$
- Any one of the formulas below can be used, using the parameterization of the shape as x and y

$$Area\ enclosed\ by\ C = \oint_C x\, dy = \oint_C -y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx$$

Additivity of Circulation: the domain D can be decomposed into two non-overlapping regions and then their circulations can be added together.

$$\oint_{\partial D} \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_{\partial D_1} \boldsymbol{F}\cdot d\boldsymbol{r} + \oint_{\partial D_2} \boldsymbol{F}\cdot d\boldsymbol{r}$$

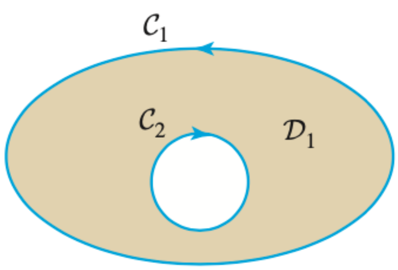


HIGHLIGHT #2: Other Forms of Green's Theorem

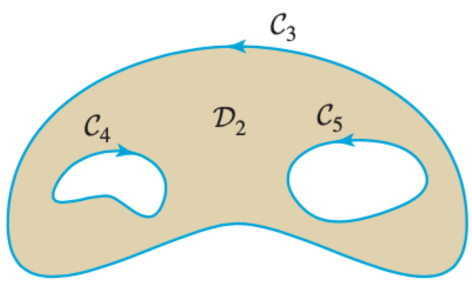
More General Form: Like we saw above the additivity, the domains of a simply closed curve can be added together, but they can also be subtracted from each other.

- The curves inside the domain that are oriented clockwise (the negative orientation) are added because they are a double negative since they are in the negative orientation and decreasing the size of the domain
- Therefore, the domain in A could also be written $D_1 = C_1 - (-C_2)$

Examples:



(A) Oriented boundary of $\mathcal{D}_1$ is $\mathcal{C}_1 + \mathcal{C}_2$.



(B) Oriented boundary of $\mathcal{D}_2$ is $\mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_5$.

Because of this property, a more general form of Green's theorem can be written,

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Curl Form: Since Green's theorem is only used in two dimensions, we can imagine it as a three- dimensional vector with no z-component so $F = \langle F_1, F_2, 0\rangle$.

When we take the curl of this vector,

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

The z-component of curl is the same as the integrand in Green's Theorem!

Therefore, Green's Theorem can also be written as:

$$\oint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F})\, dA$$

Vector Form: this can be used to calculate the flux and is derived from the Curl Form

- This is done using the normal components of **F** around the curve *C*

$$\oint_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \iint_D div(\boldsymbol{F})\, dA$$

Useful videos for further explanation of Green's Theorem!

https://www.youtube.com/watch?v=a_zdFvYXX_c

https://www.youtube.com/watch?v=kdTxN4E0vbo

21 Part Video Series:
https://www.youtube.com/watch?v=9Jb2oWOJlng&list=PLX2gX-ftPVXWZITWo7MCFbQZwdcDQBdGX

CHECK YOUR LEARNING

1. Using Green's Theorem compute the circulation of $\boldsymbol{F}(x, y) = \langle \sin x, x^2y^3 \rangle$ around the triangular path C with counterclockwise orientation. The domain is described by $0 \leq x \leq 2$, $0 \leq y \leq x$.
2. Calculate the area of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using a line integral.
3. Calculate the flux of $\boldsymbol{F}(x, y) = \langle x^3, y^3 + y \rangle$.

THINGS YOU MAY STRUGGLE WITH

1. It's common for students to try and use Green's theorem when it can't be applied. Therefore, try to relate Green's theorem to circulation, meaning it can only be used for closed two dimensional curves, like a circle. It's not a solution for all problems, but it can be a helpful one for certain situations.
2. While there are a lot of different versions of Green's Theorem they are all the same thing. It will depend on what the problem gives you that will help you determine which version will be the most useful!

That's all for this week! I hope this was a helpful review of 17.1! Feel free to visit the Tutoring Center Website for more information at www.baylor.edu/tutoring.

Answers:

1. $\frac{16}{3}$
2. $Area = \pi ab$
3. $Flux = \frac{5\pi}{2}$