Week 6
MTH-2321 - Calculus III

Hello and Welcome to the weekly resources for MTH-2321 - Calculus III!

This week is Week 6 of class, and typically in this week of the semester. your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135

Key Words: Gradient, Directional Derivatives, Optimization

TOPIC OF THE WEEK

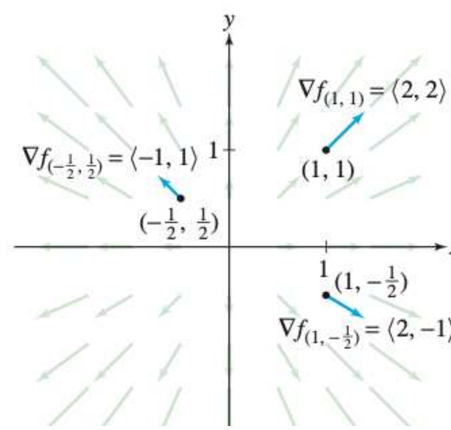
Gradients

The gradient ∇f_P of a function f is the vector containing the partial derivatives of f in each direction at point P = (a, b, c).

$$\nabla f_P = \nabla f(a,b,c) = \langle f_x(a,b,c)\,, f_y(a,b,c)\,, f_z(a,b,c)\ \rangle$$

Or it can be written without the point P:

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$



Gradient for $f = x^2 + y^2$

Important Properties:

Addition: $\nabla(f+g) = \nabla f + \nabla g$
Scalar multiple: $\nabla(cf) = c\nabla f$, c is a constant
Product Rule: $\nabla(fg) = f\nabla g + g\nabla f$

Chain Rule: $\nabla\left(F\left(f(x,y,z)\right)\right) = F'(f(x,y,z))\nabla f$

- F is a differentiable function
- Same method where you take the derivative of the outside and then the inside.

Ex. Find the gradient of $g(x,y,z) = (x^2 + y^2 + z^2)^8$

Outside — Inside

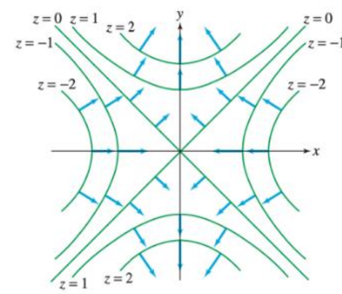
$$F = f(x,y,z)^8 \qquad and \qquad f(x,y,z) = (x^2 + y^2 + z^2)$$

$$F' = 8 * f(x,y,z)^7 \qquad \nabla f = \langle f_x, f_y, f_z \rangle$$

$$F' = 8(x^2 + y^2 + z^2)^7 \qquad \nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla g = F' * \nabla f$$

$$\boxed{\nabla g = 8(x^2 + y^2 + z^2)^7 \langle 2x, 2y, 2z \rangle}$$



Another important note: the gradient of a function is perpendicular to its level curves and points in the direction of maximum increase of the function. (see picture on the right)

HIGHLIGHT #1: CHAIN RULE FOR PATHS

A path is a curve in 3 dimensions represented by a vector-valued function $\boldsymbol{r}(t)$. Recall that if $\boldsymbol{r}(t)$ represents position, $\boldsymbol{r}'(t)$ represents velocity or a rate of change.

Chain Rule for Paths: dot product between the gradient of f and the derivative of $\boldsymbol{r}(t)$

$$\frac{d}{dt} f(\boldsymbol{r}(t)) = \nabla f_{r(t)} \cdot \boldsymbol{r}'(t)$$

$$\frac{d}{dt} f(\boldsymbol{r}(t)) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} z'(t)$$

This is used to describe how a variable f changes along the path r(t). An example is determining how the temperature changes as one travels along a path throughout the U.S (see picture)

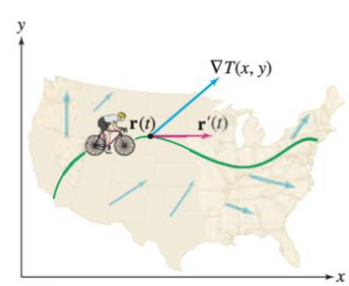


FIGURE 3 Alexa's temperature changes at the rate $\nabla T_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)$.

The chain rule can also be used in the same way for **composite functions** of two variables where x, y, and z are differentiable by the independent variables s and t.

$$f\big(x(s,t), y(s,t), z(s,t)\big)$$

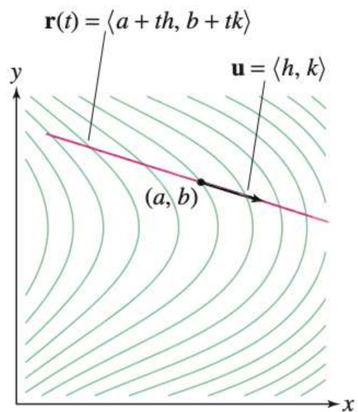
We can either take the derivative of f in terms of s or t.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

HIGHLIGHT #2: DIRECTIONAL DERIVATIVES

One application of using the chain rule for paths is directional derivatives. Consider a line through a point $P = (a, b)$ in the direction of a unit vector $\boldsymbol{u} = \langle h, k\rangle$. The parametric equations would then be:

$$\boldsymbol{r}(t) = \langle a + ht, b + kt\rangle$$



Contour map of $f(x, y)$

Directional Derivative: the derivative of $f\big(\boldsymbol{r}(t)\big)$ at $t = 0$ with respect to **u** at P.

- Notation: $D_u f(P)$ or $D_u f(a, b)$
- This represents the rate of change of f along the path represented by the point P and the unit vector **u**.

$$D_u f(P) = \nabla f_P \cdot \boldsymbol{r}'(0) = \nabla f_P \cdot \boldsymbol{u}$$

$$\boldsymbol{r}'(t) = \langle h, k\rangle = \boldsymbol{u}$$

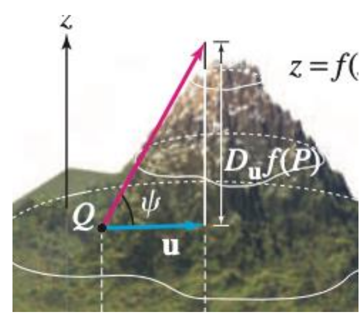
Since there is no t value, $\boldsymbol{r}'(0) = \boldsymbol{u}$

The directional derivative needs to include a unit vector, **u**. One must divide a vector by its magnitude to result in a unit vector.

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

The angle between ∇f_P and $\boldsymbol{u}$ can be solved for using the equation below. Also known as the angle between the gradient and the direction.

$$D_u f(P) = \nabla f_P \cdot \boldsymbol{u} = \|\nabla f_P\| cos\theta$$



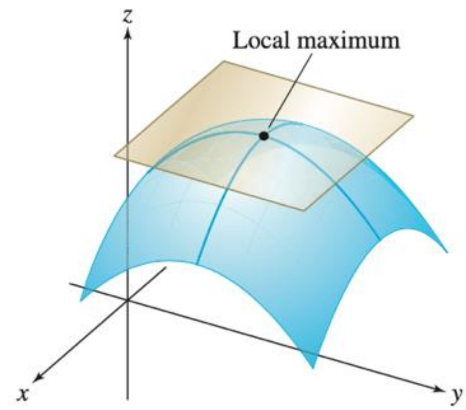
Similarly, the angle of inclination, ψ can be found. For example, the angle of inclination can be described as the angle between the ground and the side of a mountain.

$$D_u f(P) = \tan(\psi)$$

HIGHLIGHT #3: OPTIMIZATION

Optimization: process of finding the extreme values of a function, the local and global maximum and minimums

- Local: within a specified region or disk, D
- Global: within the entire domain of a function (aka absolute)



Step 1: Find critical points

- Critical points: points where the tangent plane is horizontal
- Point $P = (a, b)$ is a critical point if:

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0 \text{ or do not exist}$$

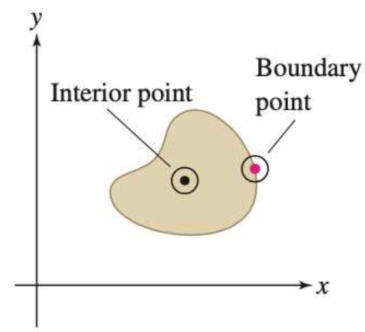
Step 2: Second Derivative Test

- D is also called the discriminate
- Determines the type of critical point using the equation:

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$$

(i) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
(ii) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(iii) If $D < 0$, then f has a saddle point at (a,b).
(iv) If $D = 0$, the test is inconclusive.

All extreme values will be local. In order to find global extrema, the interior and boundaries of the domain must be evaluated for critical points. The highest and lowest critical points will be the global extrema.



The following videos are great resources to use for additional explanation of the topics covered in this resource!

Video Series on Gradients and Directional Derivatives:
https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives#gradient-and-directional-derivatives

Chain Rule for Paths:
https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/multivariable-chain-rule/v/multivariable-chain-rule

Optimization (including global):
https://www.youtube.com/watch?v=Hg38kfK5w4E

CHECK YOUR LEARNING

1. Calculate $\nabla f_{(3,-2,4)}$ where $f = ze^{2x+3y}$.

2. Find the directional derivative in the direction of $v = \langle 2,3 \rangle$. Let $f = xe^{y}$ and $P = (2,-1)$.

3. Calculate $\frac{\partial f}{\partial s}$ where $f = xy + z$ and $x = s^2, y = st, z = t^2$.

4. Find the local extrema of $f(x^2 + y^2)e^{-x}$.

THINGS YOU MAY STRUGGLE WITH

1. A lot of things in Calculus III can be difficult to visualize since everything is shifted into 3 dimensions. Rely heavily on pictures, videos, or online 3D graphs to help be able to visualize what you are doing.

2. Most of these calculations have multiple steps are dealing with 3+ variables. Be sure to take your time and ensure you know the purpose of each variable and what to do with it. It's okay to make a chart or write out a bunch of intermediate steps to help you stay organized.

That's all for this week! I hope this was a helpful review of Chapter 14.5 - 14.7! Feel free to visit the Tutoring Center Website for more information at www.baylor.edu/tutoring.

Answers:

1. $\nabla f_{(3,-2,4)} = \langle 8,12,1 \rangle$
2. $D_u f(P) \approx 0.82$
3. $\frac{\partial f}{\partial s} = 2sy + xt$
4. (0,0) is a local min, (2,0) is saddle