Week 9
MTH-2321 - Calculus III

Hello and Welcome to the weekly resources for MTH-2321 - Calculus III!

This week is Week 9 of class, and typically in this week of the semester. your professors are covering these topics below. If you do not see the topics your particular section of class is learning this week, please take a look at other weekly resources listed on our website for additional topics throughout of the semester.

We also invite you to look at the group tutoring chart on our website to see if this course has a group tutoring session offered this semester.

If you have any questions about these study guides, group tutoring sessions, private 30 minute tutoring appointments, the Baylor Tutoring YouTube channel or any tutoring services we offer, please visit our website www.baylor.edu/tutoring or call our drop in center during open business hours. M-Th 9am-8pm on class days 254-710-4135

Key Words: Change of Variables, Vector Fields, Curl, Divergence

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TOPIC OF THE WEEK

Change of Variables

In Calculus I, you learned about <u>u-substitution</u>
- Change your function from a function of x, into a function of u

$$\int f(g(x))g'(x)dx = \int f(u)\,du$$
$$\text{Where } u = g(x) \;\; du = \; g'(x)$$

We want to do something similar for double and triple integrals!

Mapping
In this case, we change x and y into u and v, and this is often called **mapping or transformation** from one domain to another.

$$G(u.v) = (x(u.v), y(u,v))$$

Ex. Find the image of the triangle T with vertices $(1,2), (2,1), (3,4)$ under the linear map $G(u,v) = (2u - v, \ u + v)$.
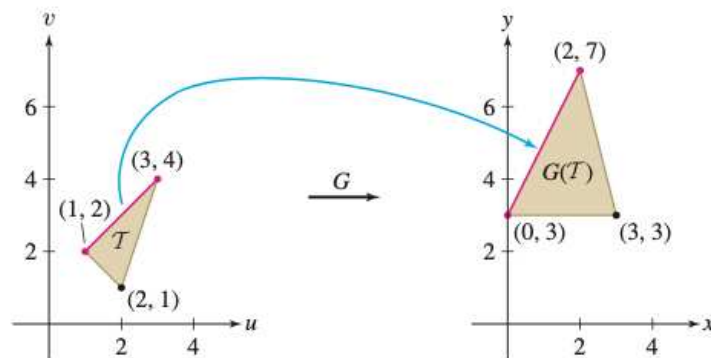
Just need to take each point and plug them into the equation where x = u and y = v.

$$G(1,2) = (2(1) - 2, 1 + 2) \qquad G(2,1) = (2(2) - 1, 2 + 1) \qquad G(3,4) = (2(3) - 4, 3 + 4)$$

$$G(1,2) = (0,3) \qquad\qquad G(2,1) = (3,3) \qquad\qquad G(3,4) = (2,7)$$

Jacobian

In order to use change of variables in a double integral we need the **Jacobian** of a map.
- Determinant calculation in terms of u and v

$$Jac(G) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- Also commonly denoted $\dfrac{\partial(x,y)}{\partial(u,v)}$

The Jacobian tells us how the area of a region changes due to map G. We multiply the Jacobian by the original area (D) and it tells us the new area given the new domain.

$$Area\big(G(D)\big) = |Jac(G)|Area(D)$$
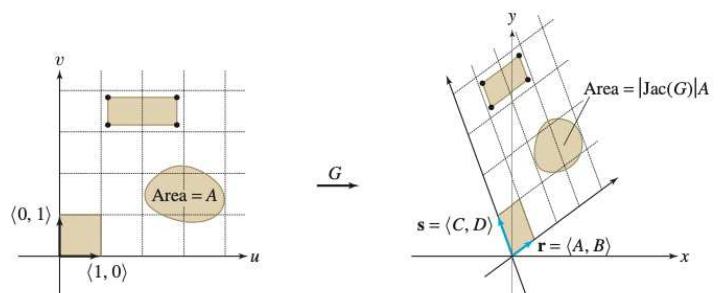
Change of Variables Formula:

- Must be one-to-one mapping from D to D_0 on the interior of the domain and f(x,y) and its partials are continuous

$$\iint_D f(x,y)dx\,dy = \iint_{D_0} f(x(u,v),y(u,v))|Jac(G)|du\,dv$$

- For three dimensions it is similar. The Jacobian Is a 3x3 matrix and the formula is:

$$\iint_W f(x,y)dx\,dy = \iint_{W_0} f(x(u,v,w),y(u,v,w),z(u,v,w))|Jac(G)|du\,dv\,dw$$

These formulas can be used to derive the change of coordinates formulas we discussed last week into cylindrical and spherical coordinates!

HIGHLIGHT #1: Vector Fields

How do we describe a region of space that is changing in all different directions? We use vector fields! They are useful for analyzing things such as wind (image to the right), electric or magnetic fields, fluid dynamics, and heat transfer which all consist of a large number of molecules moving in different potential direction and speeds.
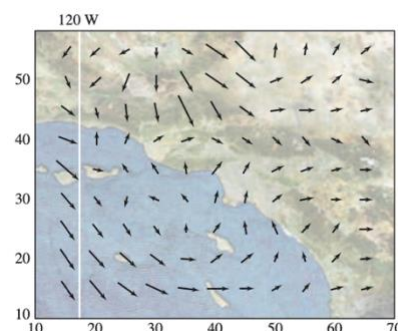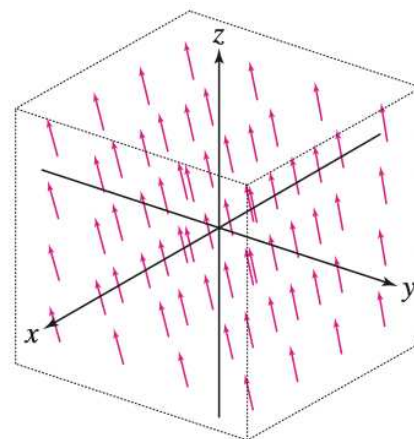

Vector Fields: a vector field **F** assigns to each point P = (x, y, z) a vector **F** (x ,y, z) that represents the speed and direction at that point.

$$F(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle = F_1 i + F_2 j$$

TYPES OF VECTOR FIELDS:

Constant Vector Field: assigns the same vector to each point (image on the right)

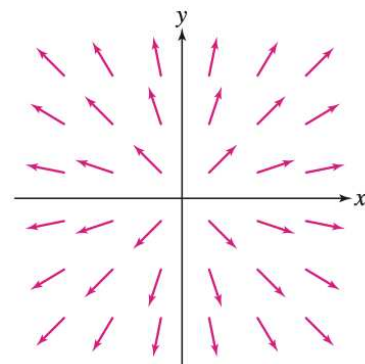Unit Vector Field: Each vector has a magnitude of 1

Radial Vector Field: the vector only depends on the distance r from P to the origin, O and is parallel to $\overrightarrow{OP}$
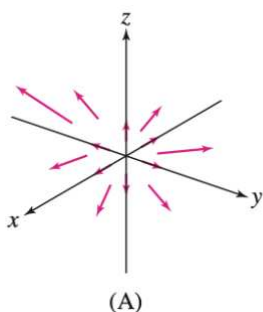- Figure on the right shows a unit radial vector field in 2D

$$e_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

The variable r is the distance formula!

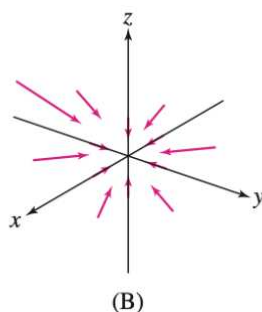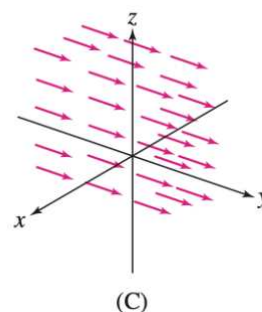# HIGHLIGHT #2: Operations on Vector Fields

DIVERGENCE: how much the flow is expanding at a certain point

(A)   (B)   (C)

$div(\boldsymbol{F}) > 0$ $div(\boldsymbol{F}) < 0$ $div(\boldsymbol{F}) = 0$

Expanding   Compressing   Neither

We calculate divergence by adding together the partial derivatives of $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$

$$div(\boldsymbol{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

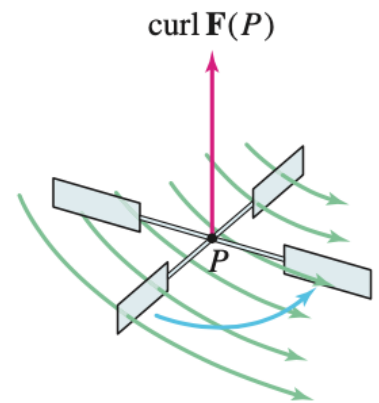This can also be written as the dot product of the gradient of **F** and **F**.

$$\nabla \cdot \boldsymbol{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

CURL: how the flow rotates, or how fast the flow would turn a paddle wheel in a fluid
- Use the right-hand rule to determine the direction of rotation
- Curl is the cross product between the gradient of **F** and **F**.

$$curl(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$curl(\mathbf{F}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$
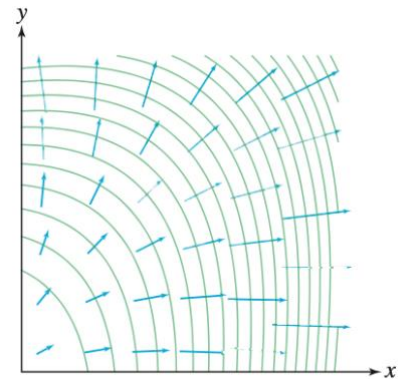
$$\text{curl } \mathbf{F}(P)$$

Conservative Vector Fields: the gradient vector field
- The function f is called a potential function for **F**
- A conservative vector field (blue) is perpendicular to the level curves (green) of the potential function f

$$\mathbf{F} = \nabla f = \left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle$$

- If a vector field is conservative, then the $curl(\mathbf{F}) = 0$ because $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

Useful videos for further explanation of change of variable and vector fields!

Change of Variable & Jacobian
https://www.youtube.com/watch?v=wUF-lyyWpUc

Vector Fields Introduction
https://www.youtube.com/watch?v=5FWAVmwMXWg

Divergence & Curl
https://www.youtube.com/watch?v=2qxxd68fZng

CHECK YOUR LEARNING

1. Integrate $f(x, y) = xy(x^2 + y^2)$ over $D: -3 \leq x^2 - y^2 \leq 3 \; and \; 1 \leq xy \leq 4$. Let u =

$x^2 - y^2$ and $v = xy$.

2. Evaluate the divergence of $F = \langle e^{xy}, xy, z^4 \rangle$ at P = (1, 0, 2).

3. Calculate the curl of $F = \langle xy, \; e^x, y + z \rangle$.

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THINGS YOU MAY STRUGGLE WITH

1. With change of variables, there are a lot of variables to remember. Work these problems slow and ensure each variable is in the appropriate space. Remember that x, y, and z should be together and u, v, w should be together. There should be no mixing of them!

2. Curl and divergence can often be difficult to visualize. For a good visual representation, watch the videos provided or find your own online! Understanding these can help you relate your answer to a graph if given and check your answer. These two properties are very similar equation wise so be sure to remember which is dot product and which is cross product.

That's all for this week! I hope this was a helpful review of 15.6 and the beginning of vector fields in Chapter 16! Feel free to visit the Tutoring Center Website for more information at www.baylor.edu/tutoring.

Answers:

1. $\frac{45}{2}$
2. $div(F) = 33$
3. $curl(F) = i + (e^x - x)k$